Math21b Solutions Week 2 Lecture 5
Section 2.2: 6,28,30,34,38,47*,36* TF14,37*

2.2.6 By Theorem 2.2.1, $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \vec{u}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

2.2.28 a D is a scaling, being of the form $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.

b E is the shear, since it is the only matrix which has the proper form (Theorem 2.2.5).

c C is the rotation, since it fits Theorem 2.2.3.

d A is the projection, following the form given in Definition 2.2.1.

e F is the reflection, using Definition 2.2.2.

2.2.30 Write $A = [\vec{v}_1 \quad \vec{v}_2]$; then $A\vec{x} = [\vec{v}_1 \quad \vec{v}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2$. We must choose $\vec{v}_1$ and $\vec{v}_2$ in such a way that $x_1\vec{v}_1 + x_2\vec{v}_2$ is a scalar multiple of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, for all x_1 and x_2. This is the case if (and only if) both $\vec{v}_1$ and $\vec{v}_2$ are scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For example, choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so that $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$.

2.2.34 Keep in mind that the columns of the matrix of a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

If T is the orthogonal projection onto a line L, then $T(\vec{x})$ will be on L for all $\vec{x}$ in $\mathbb{R}^3$; in particular, the three columns of the matrix of T will be on L, and therefore pairwise parallel. This is the case only for matrix B: B represents an orthogonal projection onto a line.

A reflection transforms orthogonal vectors into orthogonal vectors; therefore, the three columns of its matrix must be pairwise orthogonal. This is the case only for matrix E: E represents the reflection about a line.

2.2.38 a $A = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, so $\det(A) = u_1^2u_2^2 - u_1u_2u_1u_2 = 0$.

b $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, so $\det(A) = -a^2 - b^2 = -(a^2 + b^2) = -1$.

c $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, so $\det(A) = a^2 - (-b^2) = a^2 + b^2 = 1$.

d $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, both of which have determinant equal to $1^2 - 0 = 1$.

2.2.47 Write $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$.

a. $f(t) = \left(T\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\right) = \begin{bmatrix} a\cos t + b\sin t \\ c\cos t + d\sin t \end{bmatrix} \cdot \begin{bmatrix} -a\sin t + b\cos t \\ -c\sin t + d\cos t \end{bmatrix}$

$= (a\cos t + b\sin t)(-a\sin t + b\cos t) + (c\cos t + d\sin t)(-c\sin t + d\cos t)$

This function $f(t)$ is continuous, since $\cos(t)$, $\sin(t)$, and constant functions are continuous, and sums and products of continuous functions are continuous.

b. $f\left(\frac{\pi}{2}\right) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T\begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\left(T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, since T is linear.

$f(0) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The claim follows.

c. By part (b), the numbers $f(0)$ and $f\left(\frac{\pi}{2}\right)$ have different signs (one is positive and the other negative), or they are both zero. Since $f(t)$ is continuous, by part (a), we can apply the intermediate value theorem. (See Figure 2.33.)

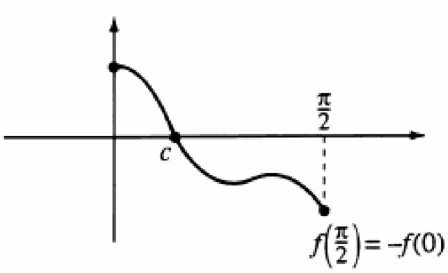

Figure 2.49: for Problem 2.2.47c.

d. Note that $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ and $\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$ are perpendicular unit vectors, for any t. If we set

2.2.36 If the vectors $\vec{v}_0, \vec{v}_1$, and $\vec{v}_2$ are defined as shown in Figure 2.28, then the parallelogram P consists of all vectors $\vec{v}$ of the form $\vec{v} = \vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2$, where $0 \le c_1, c_2 \le 1$.

The image of P consists of all vectors of the form $T(\vec{v}) = T(\vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2) = T(\vec{v}_0) + c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$.

These vectors form the parallelogram shown in Figure 2.28 on the right.

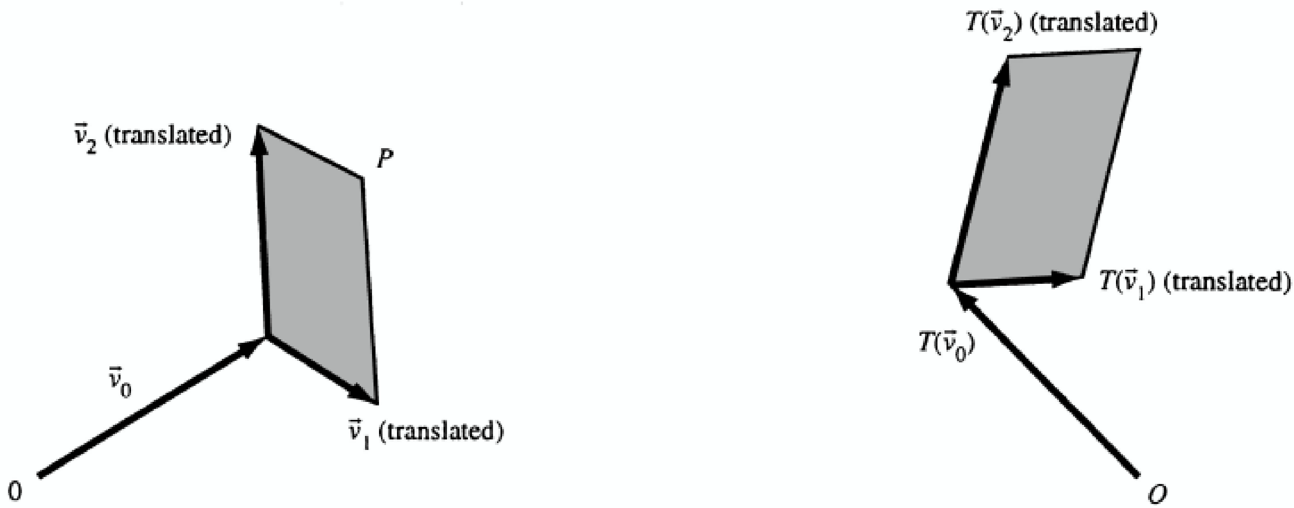

Figure 2.44: for Problem 2.2.36.

Ch 2.TF.14 T; Note that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1}$ is the unique solution.

Ch 2.TF.37 T; Note that $A^2B = AAB = ABA = BAA = BA^2$.