Math21b Solutions Week 11 Lecture 29
Section 9.3: 16,26,34,40,44,36*,20*

9.3.16 By Theorem 9.3.10, the differential equation has a particular solution of the form $f_p(t) = P\cos(t) + Q\sin(t)$. Plugging f_p into the equation we find

$(-P\cos(t) - Q\sin(t)) + 4(-P\sin(t) + Q\cos(t)) + 13(P\cos(t) + Q\sin(t)) = \cos(t)$ or

$\begin{bmatrix} 12P + 4Q = 1 \\ -4P + 12Q = 0 \end{bmatrix}$, so

$P = \frac{3}{40}$

$Q = \frac{1}{40}$.

Therefore, $f_p(t) = \frac{3}{40}\cos(t) + \frac{1}{40}\sin(t)$.

Next we find a basis of the solution space of $f''(t)+4f'(t)+13f(t) = 0$. $p_T(\lambda) = \lambda^2+4\lambda+13 = 0$ has roots $-2 \pm 3i$. By Theorem 9.3.9, $f_1(t) = e^{-2t}\cos(3t)$ and $f_2(t) = e^{-2t}\sin(3t)$ is a basis of the solution space.

By Theorem 9.3.4, the solutions of the original differential equation are of the form $f(t) = c_1 f_1(t) + c_2 f_2(t) + f_p(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + \frac{3}{40}\cos(t) + \frac{1}{40}\sin(t)$, where c_1, c_2 are arbitrary constants.

9.3.26 General solution $f(t) = c_1 e^{3t} + c_2 e^{-3t}$ (see Exercise 9), with $f'(t) = 3c_1 e^{3t} - 3c_2 e^{-3t}$

Plug in: $0 = f(0) = c_1 + c_2$ and $1 = f'(0) = 3c_1 - 3c_2$, so that $c_1 = \frac{1}{6}, c_2 = -\frac{1}{6}$, and $f(t) = \frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}$.

9.3.34 a We will take downward forces as positive.

Let g = acceleration due to gravity,

ρ = density of block

a = length of edge of block

Then (weight of block) = (mass of block) $\cdot\, g$ = (density of block)(volume of block) $g = \rho a^3 g$

buoyancy = (weight of displaced water) = (mass of displaced water) $\cdot\, g$ = (density of water) (volume of displaced water) $g = 1a^2 x(t) g = a^2 g x(t)$.

b Newton's Second Law of Motion tells us that

$m\frac{d^2x}{dt^2} = F = \text{weight} - \text{buoyancy} = \rho a^3 g - a^2 g x(t)$, where $m = \rho a^3$ is the mass of the block.

$\rho a^3 \frac{d^2x}{dt^2} = \rho a^3 g - a^2 g x(t)$

$\frac{d^2x}{dt^2} = g - \frac{g}{\rho a} x(t)$

$\frac{d^2x}{dt^2} + \frac{g}{\rho a} x = g$

constant solution $x_p = \rho a$

general solution (use Theorem 9.3.9): $x(t) = c_1 \cos\left(\sqrt{\frac{g}{\rho a}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{\rho a}}t\right) + \rho a$

Now $c_2 = 0$ since block is at rest at $t = 0$.

Plug in: $a = x(0) = c_1 + \rho a$, so that $c_1 = a - \rho a$ and

$x(t) = (a - \rho a)\cos\left(\sqrt{\frac{g}{\rho a}}t\right) + \rho a \approx 2\cos(11t) + 8$ (measured in centimeters)

c The period is $P = \frac{2\pi}{\sqrt{\frac{g}{\rho a}}} = \frac{2\pi\sqrt{\rho a}}{\sqrt{g}}$. Thus the period increases as ρ or a increases (denser wood or larger block), or as g decreases (on the moon). The period is independent of the initial state.

9.3.40 $f_T(\lambda) = \lambda^3 + \lambda^2 - \lambda - 1 = (\lambda + 1)^2(\lambda - 1) = 0$ has roots $\lambda_{1,2} = -1,\ \lambda_3 = 1$.

In other words, we can write the differential equation as $(D + 1)^2(D - 1) = 0$.

By Exercise 38, part (d), the general solution is $x(t) = e^{-t}(c_1 + c_2 t) + c_3 e^t$.

9.3.44 a Using the approach of Exercises 16 and 17 we find $x(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) - \frac{1}{40}\cos(3t) + \frac{3}{40}\sin(3t)$.

b For large $t, x(t) \approx -\frac{1}{40}\cos(3t) + \frac{3}{40}\sin(3t)$.

9.3.36 $f_T(\lambda) = \lambda^2 + 2\lambda + 101 = 0$ has roots $\lambda_{1,2} = -1 \pm 20i$.

By Theorem 9.3.9, $x(t) = e^{-t}(c_1 \cos(20t) + c_2 \sin(20t))$.

Any nonzero solution goes through the equilibrium infinitely many times. See Figure 9.47.

9.3.20 $p_T(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 1)(\lambda - 2) = 0$ has roots $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$.

By Theorem 9.3.8, the general solution is $f(t) = c_1 + c_2e^t + c_3e^{2t}$, where c_1, c_2, c_3 are arbitrary constants.