Math21b Solutions Week 11 Lecture 28
Section 4.2: 32,54,50,66,78* Section 9.3: 4 TF44*58*

4.2.32 T is not linear: $T(2) = 0 + t^2 \neq 2T(1) = 2(0 + t^2)$.

4.2.54 Use calculus to see that $T(a + bt + ct^2) = 5a + \frac{5}{2}b + \frac{35}{3}c$. The image is all of $\mathbb{R}$, so that the rank is 1. The kernel consists of all polynomials of the form $f(t) = -\frac{1}{2}b - \frac{7}{3}c + bt + ct^2$, and thus the nullity is 2.

4.2.50 Linear. $T(f(t)+g(t)) = \frac{f(t+2)+g(t+2)-f(t)-g(t)}{2} = \frac{f(t+2)-f(t)}{2} + \frac{g(t+2)-g(t)}{2} = T(f(t)) + T(g(t))$, and $T(kf(t) = \frac{kf(t+2)-kf(t)}{2} = k\frac{f(t+2)-f(t)}{2} = kT(f(t))$.

This is not an isomorphism, however, since $T(5) = \frac{5-5}{2} = 0$.

4.2.66 The kernel of T consists of all smooth functions $f(t)$ such that $T(f(t)) = f(t) - f'(t) = 0$, or $f'(t) = f(t)$. As you may recall from a discussion of exponential functions in calculus, those are the functions of the form $f(t) = Ce^t$, where C is a constant. Thus the nullity of T is 1.

9.3.4 We can look for a sinusoidal solution $x_p(t) = P\cos(3t) + Q\sin(3t)$, as in Example 7. P and Q need to be chosen in such a way that $-3P\sin(3t) + 3Q\cos(3t) - 2P\cos(3t) - 2Q\sin(3t) = \cos(3t)$ or $\begin{bmatrix} -2P + 3Q = 1 \\ -3P - 2Q = 0 \end{bmatrix}$ with solution $P = -\frac{2}{13}$ and $Q = \frac{3}{13}$. Since the general solution of $\frac{dx}{dt} - 2x = 0$ is $x(t) = Ce^{2t}$, the general solution of $\frac{dx}{dt} - 2x = \cos(3t)$ is $x(t) = Ce^{2t} - \frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)$, where C is an arbitrary constant.

4.2.78 a Check all the conditions in Definition 4.1.1. A basis is 2.

b $T(x \oplus y) = T(xy) = \ln(xy) = \ln(x) + \ln(y) = T(x) + T(y)$ and

$T(k \odot x) = T(x^k) = \ln(x^k) = k\ln(x) = kT(x)$.

The inverse of T is $L(y) = e^y$, so that T is indeed an isomorphism.

Ch 4.TF.**44** F; We can construct as many linearly independent elements in $\ker(T)$ as we want, for example, the polynomials $f(t) = t^n - \frac{1}{n+1}$, for all positive integers n.

Ch 4.TF.**58** F; If A is a scalar multiple of I_2, then all 2×2 matrices commute with A, so that the space of commuting matrices is 4 - dimensional. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ fails to be a scalar multiple of I_2, consider the equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which amounts to the system $cy - bz = 0, bx + (d-a)y - bt = 0, cx + (d-a)z - ct = 0$. If $b \neq 0$, then the first two equations are independent; if $c \neq 0$, then the first and the third equation are independent; and if $a \neq d$, then the second and the third equation are independent. Thus the rank of the system is at least two and the solution space is at most two-dimensional. (The solution space is in fact two -dimensional, since A and I_2 are independent solutions.)