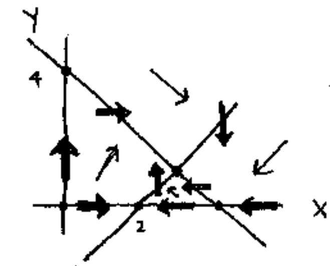
8.3 Nonlinear Systems and Linearization.

1a.



$\frac{dx}{dt} = x(2-x+y) \longrightarrow x=0, y=x-2$

$\frac{dy}{dt} = y(4-x-y) \longrightarrow y=0, y=4-x$

b. $(a,b)=(3,1)$ $\quad J=\begin{bmatrix} 2-2x+y & x \\ -y & 4-x-2y \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -1 & -1 \end{bmatrix}$

c. $f_J(\lambda) = \lambda^2+4\lambda+6 \longrightarrow \lambda_{1,2} = -2 \pm \sqrt{2}\, i$

$\det J > 0$, $\operatorname{tr} J < 0 \longrightarrow$ STABLE

2a. $\frac{dx}{dt} = x(1-x+ky-k) \longrightarrow x=0, \; x=1+ky-k$

$\frac{dy}{dt} = y(1-y+kx-k) \longrightarrow y=0, \; y=1+kx-k$

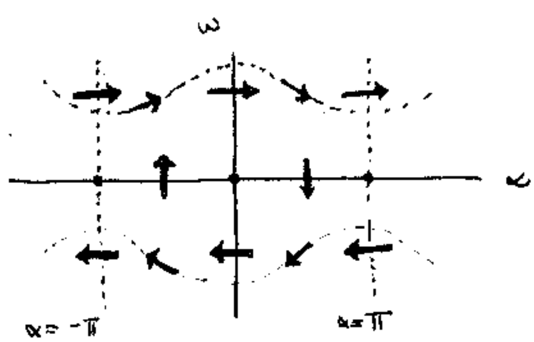
$a,b>0 \longrightarrow b = 1+k(1+kb-k)-k = 1+k^2b-k^2 \longrightarrow b=1, a=1.$

b. $J=\begin{bmatrix} 1-2x+ky-k & kx \\ ky & 1-2y+kx-k \end{bmatrix} = \begin{bmatrix} -1 & k \\ k & -1 \end{bmatrix}$

c. $f_J(\lambda) = \lambda^2+2\lambda+(1-k^2)$.

Stable if $1-k^2>0 \longrightarrow 1>k^2 \longrightarrow |k|<1$.

3a.



b. Equilibrium points at $w = 0,\ \alpha = 0, \pm\pi, \pm 2\pi, \ldots$

$$\frac{d\left(\frac{d\alpha}{dt}\right)}{d\alpha} = 0 \qquad \frac{d\left(\frac{d\alpha}{dt}\right)}{dw} = 1 \qquad \frac{d\left(\frac{dw}{dt}\right)}{d\alpha} = \frac{-g}{L}\cos\alpha \qquad \frac{d\left(\frac{dw}{dt}\right)}{dw} = 0$$

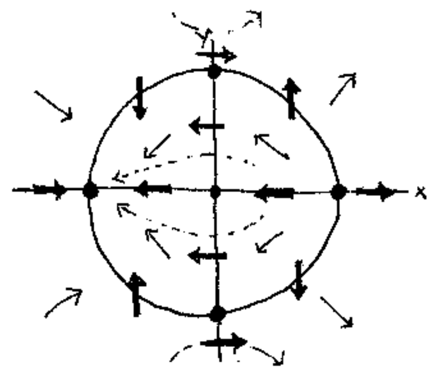
$$J = \begin{bmatrix} 0 & 1 \\ \frac{-g}{L}\cos\alpha & 0 \end{bmatrix}$$

If $\alpha = 2k\pi$, $\lambda = \pm i\sqrt{\frac{g}{L}}$ and trajectories circle the equilibrium point.

If $\alpha = (2k+1)\pi$, $\lambda = \pm\sqrt{\frac{g}{L}}$ and trajectories deviate from eq. point.

4. $\frac{dx}{dt} = x^2 + y^2 - 1 \longrightarrow x^2 + y^2 = 1$

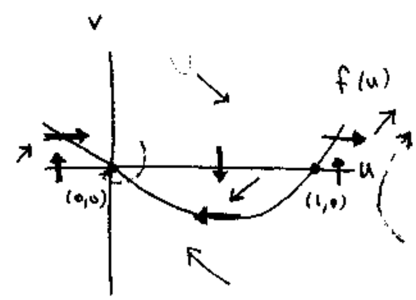
$\frac{dy}{dt} = xy \longrightarrow x = 0,\ y = 0$



Equilibrium points: $(0,1), (1,0), (0,-1), (-1,0)$

$J = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix} \longrightarrow (-1,0)$ is the only stable equilibrium

5. $\frac{du}{dt} = v \rightarrow v = 0 \qquad \frac{dv}{dt} = f(u) - v \longrightarrow f(u) = v\ (= 0)$



Equilibrium points: $(0,0), (1,0)$

$$J = \begin{bmatrix} 0 & 1 \\ f'(u) & -1 \end{bmatrix}$$

At $(1,0)$, $f'(u) > 0 \longrightarrow \lambda_1 = \frac{-1 + \sqrt{1 + 4f'(u)}}{2} > 0 \longrightarrow$ not stable

At $(0,0)$, $f'(u) < 0 \longrightarrow \lambda_1 = \frac{-1 + \sqrt{1 + 4f'(u)}}{2} < 0 \longrightarrow$ stable

(if $f'(0) < -\frac{1}{4}$, $\lambda_{1,2}$ are complex and $\text{Re}\ \lambda_{1,2} = -\frac{1}{2} < 0 \longrightarrow$ stable)