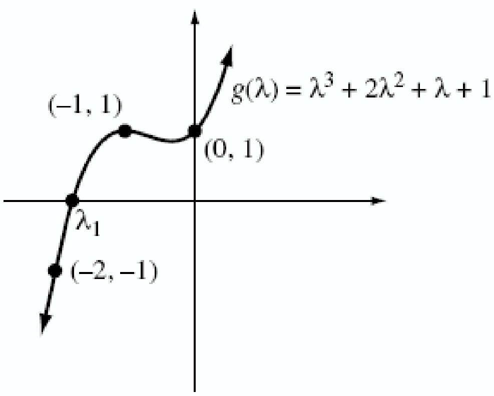
Math21b Solutions Week 10 Lecture 26
Section 9.2: 12,18,22-26,32,40,36* TF 19*

9.2.12 We will show that the real parts of all the eigenvalues are negative, so that the zero state is a stable equilibrium solution. Now the characteristic polynomial of A is $f_A(\lambda) = -\lambda^3 - 2\lambda^2 - \lambda - 1$. It is convenient to get rid of all these minus signs: The

eigenvalues are the solutions of the equation $g(\lambda) = \lambda^3 + 2\lambda^2 + \lambda + 1 = 0$. Since $g(-1) = 1$ and $g(-2) = -1$, there will be an eigenvalue λ_1 between -2 and -1. Using calculus (or a graphing calculator), we see that the equation $g(\lambda) = 0$ has no other real solutions. Thus there must be two complex conjugate eigenvalues $p \pm iq$. Now the sum of the eigenvalues is $\lambda_1 + 2p = \mathrm{tr}(A) = -2$, and $p = \frac{-2-\lambda_1}{2}$ will be negative , as claimed. The graph of $g(\lambda)$ is shown in Figure 9.33.



9.2.18 If λ_1, λ_2, λ_3 are real and negative, then $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 < 0$ and $\det(A) = \lambda_1\lambda_2\lambda_3 < 0$. If λ_1 is real and negative and $\lambda_{2,3} = p \pm iq$, where p is negative, then $\mathrm{tr}(A) = \lambda_1 + 2p < 0$ and $\det(A) = \lambda_1(p^2 + q^2) < 0$. Either way, both trace and determinant are negative.

9.2.22 $\lambda_1 = 3, \lambda_2 = 0.5; E_3 = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

System is discrete so choose VII.

9.2.23 $\lambda_{1,2} = -\frac{1}{2} \pm i, r > 1$, so that trajectory spirals outwards. Choose II.

9.2.24 $\lambda_1 = 3, \lambda_2 = 0.5, E_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. System is continuous, so choose I.

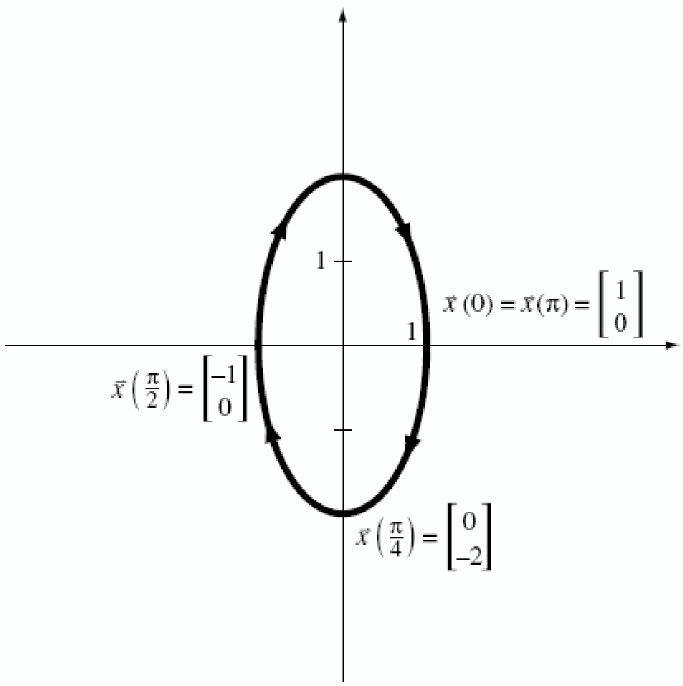
9.2.25 $\lambda_{1,2} = -\frac{1}{2} \pm i$; real part is negative so that trajectories spiral inwards in the counter-clockwise direction $\left(\text{if } \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ then } \frac{d\vec{x}}{dt} = \begin{bmatrix} -1.5 \\ 2 \end{bmatrix}\right)$. Choose IV.

9.2.26 $\lambda_1 = 1, \lambda_2 = -2; E_1 = \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_{-2} = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

System is continuous so choose V.

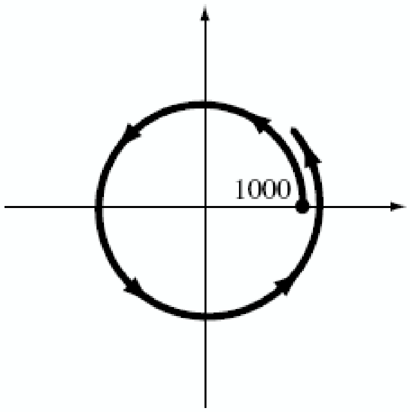
9.2.32 $\lambda_{1,2} = \pm 2i, E_{2i} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right), \vec{x}(0) = 0\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so that $a = 0$ and $b = 1$.

$\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}$. See Figure 9.37.

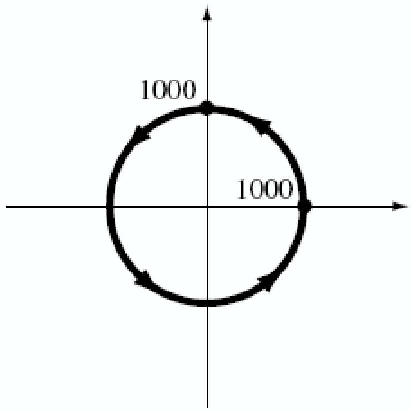


9.2.40 a $B(t) = 1000(1+0.05i)^t = 1000(r(\cos\theta + i\sin\theta))^t = 1000r^t(\cos(\theta t) + i\sin(\theta t))$, where $r = \sqrt{1+0.05^2} > 1$ and $\theta = \arctan(0.05) \approx 0.05$. See Figure 9.42.

b $B(t) = 1000e^{0.05i} = 1000(\cos(0.05t) + i\sin(0.05t))$. See Figure 9.42.



trajectory to part a
slowly spiral outwards



trajectory to part b
circle
period = $\frac{2\pi}{0.05} \cong 126$ (years)

9.2.36 a If $c = 0$ then $\lambda_{1,2} = \pm i\sqrt{b}$. The trajectories are ellipses. See Figure 9.40.

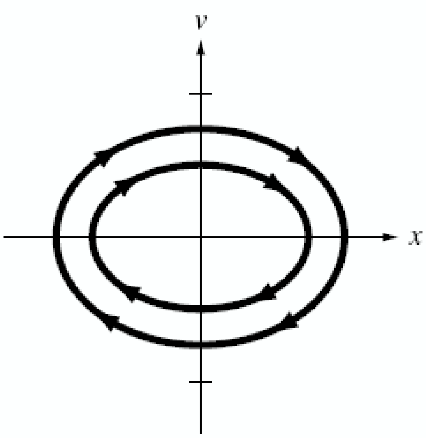


Figure 9.201: for Problem 9.2.36a.

The block *oscillates harmonically*, with period $\frac{2\pi}{\sqrt{b}}$. The zero state fails to be asymptotically stable.

b $\lambda_{1,2} = \frac{-c \pm i\sqrt{4b-c^2}}{2}$

The trajectories spiral inwards, since $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = -\frac{c}{2} < 0$. This is the case of a *damped oscillation*. The zero state is asymptotically stable. See Figure 9.41.

c This case is discussed in Exercise 9.1.55. The zero state is stable here.

9.2.19 False, consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.