Math21b Solutions Week 9 Lecture 24
Section 8.1: 4,10,24,28,42,36*,26* TF15*,16*

8.1.4 $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\2\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.10 $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 9$.

$\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}$ is in E_0 and $\vec{v}_2 = \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$ is in E_9.

Let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}}\begin{bmatrix}2\\-4\\-5\end{bmatrix}$; then $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ is an orthonormal eigenbasis.

$$S = \begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{3} & \frac{2}{3\sqrt{5}}\\ \frac{1}{\sqrt{5}} & -\frac{2}{3} & -\frac{4}{3\sqrt{5}}\\ 0 & \frac{2}{3} & -\frac{\sqrt{5}}{3}\end{bmatrix} \text{ and } D = \begin{bmatrix}0&0&0\\0&9&0\\0&0&0\end{bmatrix}.$$

8.1.24 Note that A is symmetric and orthogonal, so that the eigenvalues are 1 and -1 (see Exercise 23).

$E_1 = \text{span}\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right)$ and $E_{-1} = \text{span}\left(\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right)$, so that

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$ is an orthonormal eigenbasis.

8.1.**28** For $\lambda \neq 0$

$$f_A(\lambda) = \det \left[\begin{array}{cccc|c} -\lambda & & & 0 & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ 0 & & & -\lambda & 1 \\ 1 & 1 & \cdots & 1 & 1-\lambda \end{array}\right] = \frac{1}{\lambda} \det \left[\begin{array}{cccc|c} -\lambda & & & 0 & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ & 0 & & -\lambda & 1 \\ \lambda & \lambda & \cdots & \lambda & \lambda-\lambda^2 \end{array}\right]$$

$$= \frac{1}{\lambda} \det \left[\begin{array}{cccc|c} -\lambda & & & 0 & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ & 0 & & -\lambda & 1 \\ 0 & 0 & \cdots & 0 & -\lambda^2+\lambda+12 \end{array}\right]$$

$$= -\lambda^{11}(\lambda^2 - \lambda - 12) = -\lambda^{11}(\lambda - 4)(\lambda + 3)$$

Eigenvalues are 0 (with multiplicity 11), 4 and -3.

Eigenvalues for 0 are $\vec{e}_1 - \vec{e}_i (i = 2, \dots, 12)$,

$E_4 = \text{span} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 4 \end{bmatrix}$ (12 ones), $E_{-3} = \text{span} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ -3 \end{bmatrix}$ (12 ones)

so

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \end{bmatrix}$$

diagonalizes A, and $D = S^{-1}AS$ will have all zeros as entries except $d_{12,\ 12} = 4$ and $d_{13,\ 13} = -3$.

8.1.42 We will use the strategy outlined in Exercise 8.1.41. An orthogonal matrix that diagonalizes $A = \frac{1}{5}\begin{bmatrix} 12 & 14 \\ 14 & 33 \end{bmatrix}$ is $S = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, with $S^{-1}AS = D = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$. Now $D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = SD_1S^{-1} = \frac{1}{5}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$.

8.1.36 If $\vec{v}$ is an eigenvector with eigenvalue λ, then $\lambda\vec{v} = A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$, so that $\lambda = \lambda^2$ and therefore $\lambda = 0$ or $\lambda = 1$. Since A is symmetric, E_0 and E_1 are orthogonal complements, so that A represents the orthogonal projection onto E_1.

8.1.26 Since J_n is both orthogonal and symmetric, the eigenvalues are 1 and -1. If n is even, then both have multiplicity $\frac{n}{2}$ (as in Exercise 24). If n is odd, then the multiplicities are $\frac{n+1}{2}$ for 1 and $\frac{n-1}{2}$ for -1 (as in Exercise 25). One way to see this is to observe that $\text{tr}(J_n)$ is 0 for even n, and 1 for odd n (recall that the trace is the sum of the eigenvalues).

Ch 8.TF.15 F. Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Ch 8.TF.16 T, since AA^T is symmetric (use the spectral theorem)