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Math21b Solutions Week 9 Lecture 23
Section 7.6: 10,16,22,34,42, 38*,40*, TF*
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7.6.10 $\lambda_{1,2} = 0, \lambda_3 = 0.9$ so $\vec{0}$ is a stable equilibrium.

7.6.16 $\lambda_{1,2} = \frac{2 \pm \sqrt{1+30k}}{10}$ so $|2 \pm \sqrt{1+30k}|$ must be less than 10. $\lambda_{1,2}$ are *real* if $k \geq -\frac{1}{30}$. In this case it is required that $2 + \sqrt{1+30k} < 10$ and $-10 < 2 - \sqrt{1+30k}$, which means that $\sqrt{1+30k} < 8$ or $k < \frac{21}{10}$.

$\lambda_{1,2}$ are *complex* if $k < -\frac{1}{30}$. Here it is required that $4 + (-1 - 30k) < 100$ or $k > -\frac{97}{30}$. Overall, $\vec{0}$ is a stable equilibrium if $-\frac{97}{30} < k < \frac{21}{10}$.

7.6.22 $\lambda_{1,2} = -2 \pm 3i, r = \sqrt{13}, \theta \approx 2.16$ (in second quadrant)

$$[\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ so } \vec{x}(t) = \sqrt{13}^t \begin{bmatrix} -5\sin(\theta t) \\ \cos(\theta t) - 3\sin(\theta t) \end{bmatrix}, \text{ where } \theta \approx 2.16.$$

Spirals outwards, as in Figure 7.39.

7.6.34 a If $|\det A| = |\lambda_1 \lambda_2 \cdots \lambda_n| = |\lambda_1 \lambda_2| \cdots |\lambda_n| > 1$ then at least one eigenvalue is greater than one in modulus and the zero state fails to be stable.

b If $|\det A| = |\lambda_1||\lambda_2| \cdots |\lambda_n| < 1$ we cannot conclude anything about the stability of $\vec{0}$.

$|2||0.1| < 1$ and $|0.2||0.1| < 1$ but in the first case we would not have stability, in the second case we would.

7.6.42 a $x(t+1) = x(t) - ky(t)$

$$y(t+1) = kx(t) + y(t) = kx(t) + (1-k^2)y(t) \text{ so } \begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

b $f_A(\lambda) = \lambda^2 - (2-k^2)\lambda + 1 = 0$

The discriminant is $(2-k^2)^2 - 4 = -4k^2 + k^4 = k^2(k^2-4)$, which is negative if k is a small positive number ($k < 2$). Therefore, the eigenvalues are complex. By Theorem 7.6.4 the trajectory will be an ellipse, since $\det(A) = 1$.

7.6.38 a $T(\vec{v}) = A\vec{v} + \vec{b} = \vec{v}$ if $\vec{v} - A\vec{v} = \vec{b}$ or $(I_n - A)\vec{v} = \vec{b}$.

$I_n - A$ is invertible since 1 is not an eigenvalue of A. Therefore, $\vec{v} = (I_n - A)^{-1}\,\vec{b}$ is the only solution.

b Let $\vec{y}(t) = \vec{x}(t) - \vec{v}$ be the deviation of $\vec{x}(t)$ from the equilibrium $\vec{v}$.

Then $\vec{y}(t+1) = \vec{x}(t+1) - \vec{v} = A\vec{x}(t) + \vec{b} - \vec{v} = A(\vec{y}(t) + \vec{v}) + \vec{b} - \vec{v} = A\vec{y}(t) + A\vec{v} + \vec{b} - \vec{v} = A\vec{y}(t)$, so that $\vec{y}(t) = A^t\vec{y}(0)$, or $\vec{x}(t) = \vec{v} + A^t(\vec{x}_0 - \vec{v})$.

$\lim_{t\to\infty} \vec{x}(t) = \vec{v}$ for all $\vec{x}_0$ if $\lim_{t\to\infty} A^t(\vec{x}_0 - \vec{v}) = \vec{0}$. This is the case if the modulus of all the eigenvalues of A is less than 1.

7.6.40 a $A^TA = \begin{bmatrix} B^T & C^T \\ -C & B \end{bmatrix} \begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix} = (p^2 + q^2 + r^2 + s^2)I_4$

b By part a, $A^{-1} = \frac{1}{p^2+q^2+r^2+s^2} A^T$ if $A \neq 0$.

c $(\det A)^2 = (p^2 + q^2 + r^2 + s^2)^4$, by part a, so that $\det A = \pm(p^2 + q^2 + r^2 + s^2)^2$.

Laplace Expansion along the first row produces the term $+p^4$,

so that $\det(A) = (p^2 + q^2 + r^2 + s^2)^2$.

d Consider $\det(A - \lambda I_4)$. Note that the matrix $A - \lambda I_4$ has the same "format" as A, with p replaced by $p - \lambda$ and q, r, s remaining unchanged. By part c, $\det(A - \lambda I_4) = ((p - \lambda)^2 + q^2 + r^2 + s^2)^2 = 0$ when

$(p - \lambda)^2 = -q^2 - r^2 - s^2$

$p - \lambda = \pm i\sqrt{q^2 + r^2 + s^2}$

$\lambda = p \pm i\sqrt{q^2 + r^2 + s^2}$

Each of these eigenvalues has algebraic multiplicity 2 (if $q = r = s = 0$ then $\lambda = p$ has algebraic multiplicity 4).