Math21b Solutions Week 8 Lecture 22

7.5: 2(but for z8),14,26,32,38,50*,36* TF11*,41*

7.5.3 If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

$z^n = 1$ if $r = 1, \cos(n\theta) = 1, \sin(n\theta) = 0$ so $n\theta = 2k\pi$ for an integer k, and $\theta = \frac{2k\pi}{n}$,

i.e. $z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$, $k = 0, 1, 2, \ldots, n-1$. See Figure 7.26.

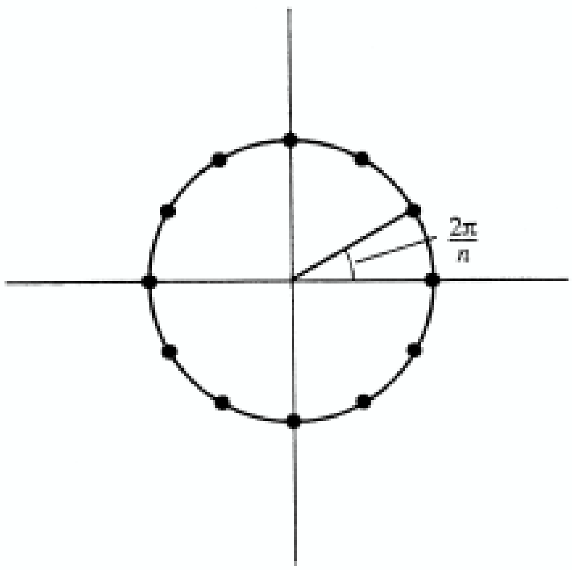

Figure 7.130: for Problem 7.5.3.

For $z^8 = 1$, $z = \cos\left(\frac{2k\pi}{8}\right) + i\sin\left(\frac{2k\pi}{8}\right)$ for n = 0,1,2,3,4,5,6,7.

Thus $z = 1, \frac{1}{\sqrt{2}}(1+i), i, \frac{1}{\sqrt{2}}(-1+i), -1, \frac{1}{\sqrt{2}}(-1-i), -i, \frac{1}{\sqrt{2}}(1-i)$

7.5.14 The eigenvalues are $\pm i$. We have $a = 0, b = 1$. The matrix is conjugated to $S^{-1}AS = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with $S = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}$, which contains the eigenvectors of A as columns.

7.5.26 $f_A(\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - 2\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda - 2)\lambda = 0$, so $\lambda_{1,2} = 1 \pm i, \lambda_3 = 2, \lambda_4 = 0$.

7.5.32 a $\vec{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} 0.6a(t) + 0.1m(t) + 0.5s(t) \\ 0.2a(t) + 0.7m(t) + 0.1s(t) \\ 0.2a(t) + 0.2m(t) + 0.4s(t) \end{bmatrix}$ so $A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$.

Note that A is a regular transition matrix.

b By Exercise 30, $\lim_{t\to\infty}(A^t) = [\vec{v}\,\vec{v}\,\vec{v}]$, where $\vec{v}$ is the unique eigenvector of A with eigenvalue 1 and column sum 1. We find that $\vec{v} = \begin{bmatrix} 0.4 \\ 0.35 \\ 0.25 \end{bmatrix}$.

Now $\lim_{t\to\infty} \vec{x}(t) = \lim_{t\to\infty}(A^t\vec{x}_0) = \left(\lim_{t\to\infty} A^t\right)\vec{x}_0 = [\vec{v}\,\vec{v}\,\vec{v}]\vec{x}_0 = \vec{v}$, since the components of $\vec{x}_0$ add up to 1. The market shares approach 40%, 35%, and 25%, respectively, regardless of the initial shares.

7.5.38 a $C_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C_4^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, C_4^4 = I_4$, then $C_4^{4+k} = C_4^k$.

Figure 7.31 illustrates how C_4 acts on the basis vectors $\vec{e}_i$.

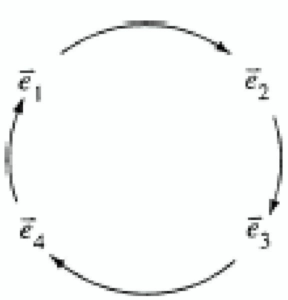

Figure 7.135: for Problem 7.5.38a.

b The eigenvalues are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i$, and $\lambda_4 = -i$, and for each eigenvalue $\lambda_k, \vec{v}_k = \begin{bmatrix} \lambda_k^3 \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix}$ is an associated eigenvector.

c $M = aI_4 + bC_4 + cC_4^2 + dC_4^3$

If $\vec{v}$ is an eigenvector of C_4 with eigenvalue λ, then $M\vec{v} = a\vec{v} + b\lambda\vec{v} + c\lambda^2\vec{v} + d\lambda^3\vec{v} = (a + b\lambda + c\lambda^2 + d\lambda^3)\vec{v}$, so that $\vec{v}$ is an eigenvector of M as well, with eigenvalue $a + b\lambda + c\lambda^2 + d\lambda^3$.

The eigenbasis for C_4 we found in part b is an eigenbasis for all circulant 4×4 matrices.

7.5.50 The eigenvalues are 0, 0, 1. Since the kernel is always two-dimensional, with basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, the matrix is diagonalizable for all values of constant a.

7.5.36 a The entries in the first row are age-specific birth rates and the entries just below the diagonal are age-specific survival rates. For example, the entry 1.6 in the first row tells us that during the next 15 years the people who are 15–30 years old today will *on average* have 1.6 children (3.2 per couple) who will survive to the next census. The entry 0.53 tells us that 53% of those in the age group 45–60 today will still be alive in 15 years (they will then be in the age group 60–75).

b Using technology, we find the largest eigenvalue $\lambda_1 = 1.908$ with associated eigenvector

$$\vec{v}_1 \approx \begin{bmatrix} 0.574 \\ 0.247 \\ 0.115 \\ 0.047 \\ 0.014 \\ 0.002 \end{bmatrix}.$$

The components of $\vec{v}_1$ give the distribution of the population among the age groups in the long run, assuming that current trends continue. λ_1 gives the factor by which the population will grow in the long run in a period of 15 years; this translates to an annual growth factor of $\sqrt[15]{1.908} \approx 1.044$, or an annual growth of about 4.4%.

Ch 7.TF.**11** T, by Example 6 of Section 7.5

Ch **7**.TF.**41** F; Consider a rotation through $\pi/2$.