Math21b Solutions Week 7 Lecture 19
Section 7.1: 10,36 and 7.2: 12,28,40,25*,26* TF6*,20*

7.1.10 We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, i.e. the desired matrices must have the form $\begin{bmatrix} 5-2b & b \\ 10-2d & d \end{bmatrix}$.

7.1.36 We want A such that $A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, i.e. $A\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}$, so

$$A = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}.$$

7.2.12 $f_A(\lambda) = \lambda(\lambda+1)(\lambda-1)^2$ so $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 1$ (Algebraic multiplicity 2).

7.2.28 a $w(t+1) = 0.8w(t) + 0.1m(t)$

$m(t+1) = 0.2w(t) + 0.9m(t)$

so $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ which is a regular transition matrix since its columns sum to 1 and its entries are positive.

b The eigenvectors of A are $\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_1 = 1$, and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $\lambda_2 = 0.7$.

$\vec{x}_0 = \begin{bmatrix} 1200 \\ 0 \end{bmatrix} = 400\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $\vec{x}(t) = 400\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800(0.7)^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or

$w(t) = 400 + 800(0.7)^t$

$m(t) = 800 - 800(0.7)^t$.

c As $t \to \infty$, $w(t) \to 400$ so Wipfs won't have to close the store.

7.2.40 Let the entries of A be a_{ij} and the entries of B be b_{ij}.

Now, $\text{tr}(AB) = (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + \cdots + a_{2n}b_{n2}) + \cdots + (a_{n1}b_{1n} + \cdots + a_{nn}b_{nn})$. This is the sum of all products of the form $a_{ij}b_{ji}$.

We see that $\text{tr}(BA) = (b_{11}a_{11} + \cdots + b_{1n}a_{n1}) + \cdots + (b_{n1}a_{1n} + \cdots + b_{nn}a_{nn})$, which also is the sum of all products of the form $b_{ji}a_{ij} = a_{ij}b_{ji}$. Thus, $\text{tr}(AB) = \text{tr}(BA)$.

7.2.25 $A\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} ab + cb \\ cb + cd \end{bmatrix} = \begin{bmatrix} (a+c)b \\ (b+d)c \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$ since $a + c = b + d = 1$; therefore, $\begin{bmatrix} b \\ c \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

Also, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a - b \\ c - d \end{bmatrix} = (a-b)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ since $a - b = -(c - d)$; therefore, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = a - b$. Note that $|a - b| < 1$; a possible phase portrait is shown in Figure 7.11.

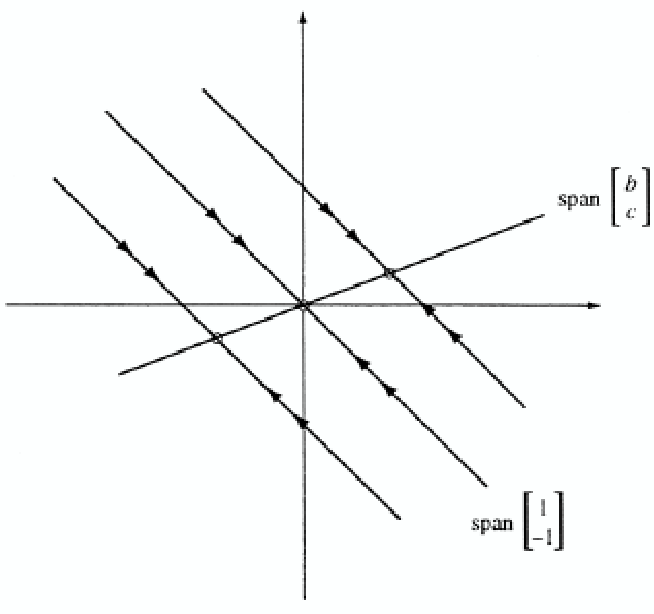

Figure 7.116: for Problem 7.2.25.

7.2.26 Here $\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$ with $\lambda_1 = 1$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $\lambda_2 = a - b = 0.25$. See Figure 7.12.

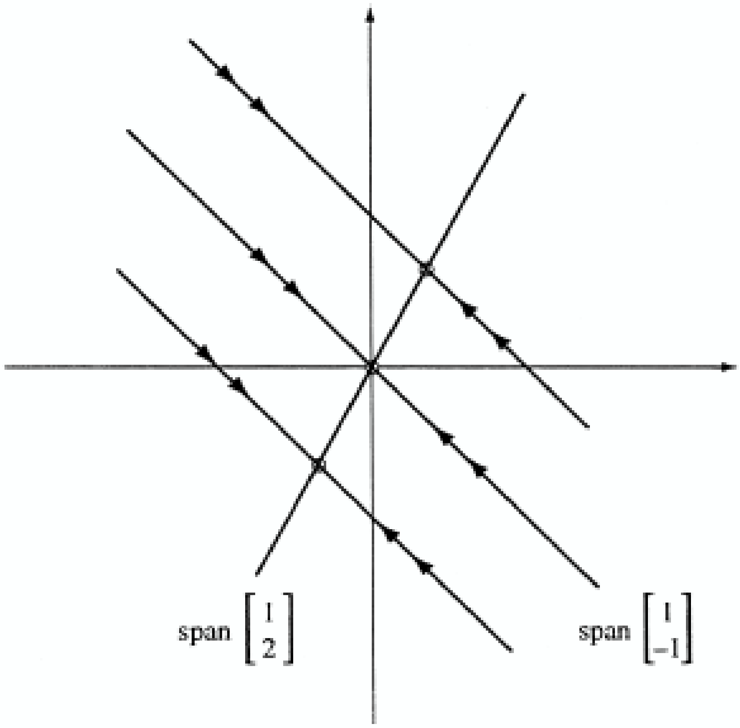

Figure 7.117: for Problem 7.2.26.

Ch 7.TF.6 F, by Theorem 7.2.7.

Ch 7.TF.20 F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, for example.