Math21b Solutions Week 7 Lecture 17
Section 6.1: 40,42,48,54,56,45*,60* TF4*,30*

6.1.40 There is one pattern with a nonzero product. It contains all the nonzero entries of the matrix, with seven inversions. Therefore, the determinant is $= -120$.

6.1.42 $\det(\begin{bmatrix} 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 9 & 7 & 9 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 3 & 4 & 5 & 8 & 5 \end{bmatrix}) = 540$. There is only one pattern which has a nonzero product, $(a_{13}, a_{24}, a_{32}, a_{45}, a_{51}) = (2, 2, 9, 5, 3)$. There are six inversions and its signature is therefore 1. The determinant is $3 * 9 * 2 * 2 * 5 = 540$.

6.1.48 Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $\det(A) = \det(B) = \det(C) = \det(D) = 0$ hence $\det(A)\det(D) - \det(B)\det(C) = 0$ but $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = -1$.

6.1.54 The determinant of the matrix

$$A = \begin{bmatrix} 1 & 1000 & 2 & 3 & 4 \\ 5 & 6 & 7 & 1000 & 8 \\ 1000 & 9 & 8 & 7 & 6 \\ 1 & 2 & 1000 & 3 & 4 \end{bmatrix}$$

is positive, because there is a dominant pattern containing all the 1000's with 4 inversions. This pattern contributes $(1000)^5 = 10^{15}$ to the determinant. There are $5! - 1 = 119$ other patterns with at most 3 entries being 1000, the others being ≤ 9. Thus the product associated with each of those pattern is less than $(1000)^3(10)^2 = 11^{11}$. Now $\det(A) > 10^{15} - 119 * 10^{11} > 0$.

6.1.56 a $\det(M_4) = \det(M_5) = 1$. $\det(M_2) = \det(M_3) = \det(M_6) = \det(M_7) = -1$.

b $d_1 = 1, d_2 = -1, d_3 = -1, d_4 = 1, d_5 = 1, d_6 = -1, d_7 = -1, d_8 = 1$. We notice an oscillation between -1 and 1 with every other increase and that $d_{n+4} = d_n$. Lets prove this: there is only one pattern with a nonzero product, containing all the 1's. The number of inversions is $(n-1)+(n-2)+\ldots+2+1 = \sum_{k=1}^{n-1} k = n(n-1)/2$. This number is even if either n or $n-1$ is divisble by 4, that is, for $n = 4, 5, 8, 9, 12, 13, \ldots$. We can write $\det(M_n) = (-1)^{n(n-1)/2}$.

c By the periodicity in part b, we see that d_{100} will be equal to $d_{100-24(4)} = d_4 = 1$.

6.1.45 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A^T) = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb = \det(A)$. It turns out that $\det(A^T) = \det(A)$.

6.1.60 a This function is linear in both columns and rows. It it is not alternating in the columns.

b This function is linear on both columns, but not linear in rows and is not alternating on the columns.
We have linearity in the first column since cd is a linear combination of the entries a and c in the first column. However, it fails to be linear in the second row. Multipying the row by 2 for example changes the value by 4.

c The function is not linear in both columns, but linear in columns and not alternating in the columns.

d This function is $-\det(A)$. It is linear in both columns and and both rows and alternating in columns.

e This function is linear both in rows and columns and not alternating in columns.

Ch 6.TF.4 T; We have $\det(-A) = (-1)^6 \det(A) = \det(A)$, by Theorem 6.2.3a.

Ch 6.TF.30 F; Let $A = 2I_2$, for example