Math21b Solutions Week 6 Lecture 16
Section 5.4: 4,10,24,34,40,20*,18* TF24*,18*

5.4.4 By Theorem 5.4.1, the equation $(\text{im } B)^\perp = \ker(B^T)$ holds for any matrix B. Now let $B = A^T$. Then $(\text{im}(A^T))^\perp = \ker(A)$. Taking transposes of both sides and using Theorem 5.1.8d we obtain $\text{im}(A^T) = (\ker A)^\perp$, as claimed.

5.4.10 a If $\vec{x}$ is an arbitrary solution of the system $A\vec{x} = \vec{b}$, let $\vec{x}_h = \text{proj}_V \vec{x}$, where $V = \ker(A)$, and $\vec{x}_0 = \vec{x} - \text{proj}_V \vec{x}$. Note that $\vec{b} = A\vec{x} = A(\vec{x}_h + \vec{x}_0) = A\vec{x}_h + A\vec{x}_0 = A\vec{x}_0$, since $\vec{x}_h$ is in $\ker(A)$.

b If $\vec{x}_0$ and $\vec{x}_1$ are two solutions of the system $A\vec{x} = \vec{b}$, both from $(\ker A)^\perp$, then $\vec{x}_1 - \vec{x}_0$ is in the subspace $(\ker A)^\perp$ as well. Also, $A(\vec{x}_1 - \vec{x}_0) = A\vec{x}_1 - A\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}$, so that $\vec{x}_1 - \vec{x}_0$ is in $\ker(A)$. By Theorem 5.1.8b, it follows that $\vec{x}_1 - \vec{x}_0 = \vec{0}$, or $\vec{x}_1 = \vec{x}_0$, as claimed.

c Write $\vec{x}_1 = \vec{x}_h + \vec{x}_0$ as in part a; note that $\vec{x}_h$ is orthogonal to $\vec{x}_0$. The claim now follows from the Pythagorean Theorem (Theorem 5.1.9).

5.4.24 Using Theorem 5.4.6, we find $\vec{x}^* = [2]$.

5.4.34 We want $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & \sin(0.5) & \cos(0.5) & \sin(1) & \cos(1) \\ 1 & \sin(1) & \cos(1) & \sin(2) & \cos(2) \\ 1 & \sin(1.5) & \cos(1.5) & \sin(3) & \cos(3) \\ 1 & \sin(2) & \cos(2) & \sin(4) & \cos(4) \\ 1 & \sin(2.5) & \cos(2.5) & \sin(5) & \cos(5) \\ 1 & \sin(3) & \cos(3) & \sin(6) & \cos(6) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 1 \\ 1.5 \\ 2 \\ 2.5 \\ 3 \end{bmatrix}$$

Since the columns of the coefficient matrix are linearly independent, its kernel is $\{\vec{0}\}$. We can use Theorem 5.4.6 to compute $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \approx \begin{bmatrix} 1.5 \\ 0.109 \\ -1.537 \\ 0.303 \\ 0.043 \end{bmatrix}$ so $f^*(t) \approx 1.5 + 0.109\sin(t) - 1.537\cos(t) + 0.303\sin(2t) + 0.043\cos(2t)$.

5.4.40 First we look for $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ such that $\log D = c_0 + c_1 \log a$.

Proceeding as in Exercise 39, we get $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* \approx \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$, i.e. $\log D \approx 1.5 \log a$, hence $D \approx 10^{1.5\log a} = a^{1.5}$.

Note that the formula $D = a^{1.5}$ is Kepler's third law of planetary motion.

5.4.20 Using Theorem 5.4.6, we find $\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Note that $\vec{b} - A\vec{x}^*$ is perpendicular to the two columns of A.

5.4.18 Yes! By Exercise 17, $\operatorname{rank}(A) = \operatorname{rank}(A^TA)$. Substituting A^T for A in Exercise 17 and using Theorem 5.3.9c, we find that $\operatorname{rank}(A) = \operatorname{rank}(A^T) = \operatorname{rank}(AA^T)$. The claim follows.

Ch 5.TF.24 F. Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, or any other symmetric matrix that fails to be orthogonal.

Ch 5.TF.18 T. Consider the QR factorization (Theorem 5.2.2)