5.1.8 Since $\vec{u} \cdot \vec{v} = 4 - 24 + 20 = 0$, the two vectors enclose a right angle.

5.1.16 You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors $\vec{x}$ that are orthogonal to $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, then we identify the unit vectors among them.

Finding the vectors $\vec{x}$ with $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \vec{x} \cdot \vec{v}_3 = 0$ amounts to solving the system

$$\begin{bmatrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{bmatrix}$$

$\left(\text{we can omit all the coefficients } \frac{1}{2}\right)$.

The solutions are of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$.

Since $\|\vec{x}\| = 2|t|$, we have a unit vector if $t = \frac{1}{2}$ or $t = -\frac{1}{2}$. Thus there are two possible choices for $\vec{v}_4$:

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

5.1.18 a $\|\vec{x}\|^2 = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$ (use the formula for a geometric series, with $a = \frac{1}{4}$), so that $\|\vec{x}\| = \frac{2}{\sqrt{3}} \approx 1.155$.

b If we let $\vec{u} = (1, 0, 0, \ldots)$ and $\vec{v} = \left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$, then

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \arccos \frac{1}{\frac{2}{\sqrt{3}}} = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6} (= 30°).$$

c $\vec{x} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots, \frac{1}{\sqrt{n}}, \cdots\right)$ does the job, since the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges (a fact discussed in introductory calculus classes).

d If we let $\vec{v} = (1, 0, 0, \ldots)$, $\vec{x} = \left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$ and $\vec{u} = \frac{\vec{x}}{\|\vec{x}\|} = \frac{\sqrt{3}}{2}\left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$ then

$$\text{proj}_L \vec{v} = (\vec{u} \cdot \vec{v})\vec{u} = \frac{3}{4}\left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right).$$

5.1.20 On the line L spanned by $\vec{x}$ we want to find the vector $m\vec{x}$ closest to $\vec{y}$ (that is, we want $\|m\vec{x} - \vec{y}\|$ to be minimal). We want $m\vec{x} - \vec{y}$ to be perpendicular to L (that is, to $\vec{x}$), which means that $\vec{x} \cdot (m\vec{x} - \vec{y}) = 0$ or $m(\vec{x} \cdot \vec{x}) - \vec{x} \cdot \vec{y} = 0$ or $m = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}} \approx \frac{4182.9}{198.53^2} \approx 0.106$.

Recall that the correlation coefficient r is $r = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|\|\vec{y}\|}$, so that $m = \frac{\|\vec{y}\|}{\|\vec{x}\|} r$. See Figure 5.3.
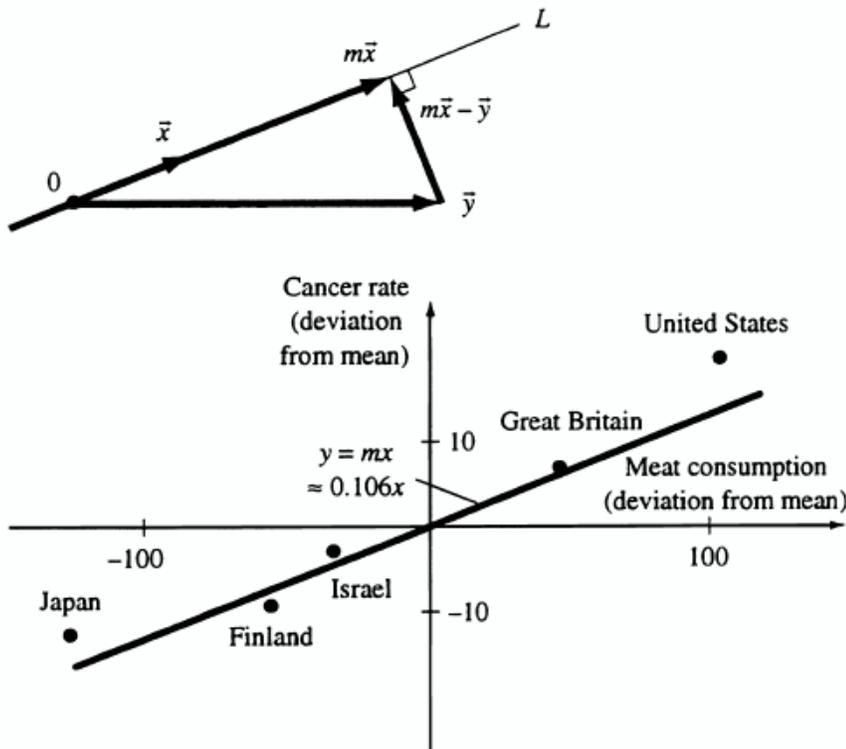

Figure 5.75: for Problem 5.1.20.

5.1.28 Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{u}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Now we can use Theorem 5.1.5, with $\vec{x} = \vec{e}_1 : \operatorname{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + (\vec{u}_3 \cdot \vec{x})\vec{u}_3 =$

$$\frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

5.1.38 Since $\vec{v}_1$ and $\vec{v}_2$ are unit vectors, the condition $\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\|\|\vec{v}_2\| \cos(\alpha) = \cos(\alpha) = \frac{1}{2}$ implies that $\vec{v}_1$ and $\vec{v}_2$ enclose an angle of $60°$ $\left(= \frac{\pi}{3}\right)$. The vectors $\vec{v}_1$ and $\vec{v}_3$ enclose an angle of $60°$ as well.

In the case $n = 2$ there are two possible scenarios: either $\vec{v}_2 = \vec{v}_3$, or $\vec{v}_2$ and $\vec{v}_3$ enclose an angle of $120°$. Therefore, either $\vec{v}_2 \cdot \vec{v}_3 = 1$ or $\vec{v}_2 \cdot \vec{v}_3 = \cos(120°) = -\frac{1}{2}$.

In the case $n = 3$, the vectors $\vec{v}_2$ and $\vec{v}_3$ could enclose any angle between $0°$ (if $\vec{v}_2 = \vec{v}_3$) and $120°$, as illustrated in Figure 5.7. We have $-\frac{1}{2} \leq \vec{v}_2 \cdot \vec{v}_3 \leq 1$.
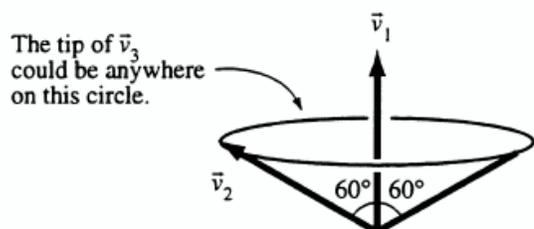

Figure 5.79: for Problem 5.1.38.

For example, consider $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right) \cos\theta \\ \left(\frac{\sqrt{3}}{2}\right) \sin\theta \\ \frac{1}{2} \end{bmatrix}$

Note that $\vec{v}_2 \cdot \vec{v}_3 = \left(\frac{3}{4}\right) \sin\theta + \frac{1}{4}$ could be anything between $-\frac{1}{2}$ (when $\sin\theta = -1$) and 1 (when $\sin\theta = 1$), as claimed.

If n exceeds three, we can consider the orthogonal projection $\vec{w}$ of $\vec{v}_3$ onto the plane E spanned by $\vec{v}_1$ and $\vec{v}_2$.

Since $\text{proj}_{\vec{v}_1} \vec{w} = (\vec{v}_1 \cdot \vec{w})\vec{v}_1 = \frac{1}{2}\vec{v}_1$, and since $\|\vec{w}\| \leq \|\vec{v}_3\| = 1$, (by Theorem 5.1.10), the tip of $\vec{w}$ will be on the line segment in Figure 5.8. Note that the angle ϕ enclosed by the vectors $\vec{v}_2$ and $\vec{w}$ is between $0°$ and $120°$, so that $\cos\phi$ is between $-\frac{1}{2}$ and 1.

Therefore, $\vec{v}_2 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{w} = \|\vec{w}\| \cos\phi$ is between $-\frac{1}{2}$ and 1.

This implies that $\angle(\vec{v}_2, \vec{v}_3)$ is between $0°$ and $120°$ as well. To see that all these values are attained, add $(n - 3)$ zeros to the three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$ given above.
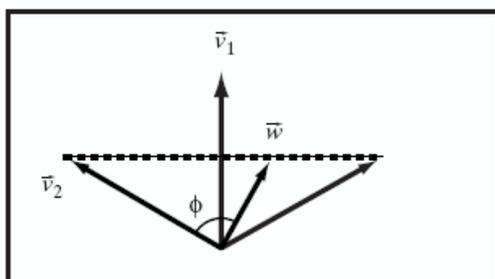

Figure 5.80: for Problem 5.1.38.

5.1.14 The horizontal components of $\vec{F}_1$ and $\vec{F}_2$ are $-\|\vec{F}_1\|\sin\beta$ and $\|\vec{F}_2\|\sin\alpha$, respectively (the horizontal component of $\vec{F}_3$ is zero).

Since the system is at rest, the horizontal components must add up to 0, so that $-\|\vec{F}_1\|\sin\beta + \|\vec{F}_2\|\sin\alpha = 0$ or $\|\vec{F}_1\|\sin\beta = \|\vec{F}_2\|\sin\alpha$ or $\frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} = \frac{\sin\alpha}{\sin\beta}$.

To find $\frac{\overline{EA}}{\overline{EB}}$, note that $\overline{EA} = \overline{ED}\tan\alpha$ and $\overline{EB} = \overline{ED}\tan\beta$ so that $\frac{\overline{EA}}{\overline{EB}} = \frac{\tan\alpha}{\tan\beta} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{\cos\beta}{\cos\alpha} = \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|}\frac{\cos\beta}{\cos\alpha}$. Since α and β are two distinct acute angles, it follows that $\frac{\overline{EA}}{\overline{EB}} \neq \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|}$, so that Leonardo was mistaken.

Ch 5.TF.5 F. The correct formula is $\operatorname{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$, by Definition 2.2.1.

Ch 5.TF.14 F. It is required that the columns of A be orthonormal (Theorem 5.3.10). As a counterexample, consider $A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, with $AA^T = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$.