Math21b Solutions Week 5 Lecture 12
4.1: 1-5,14,34,44,52,42*,56* TF18/30*

4.1.1 Not a subspace since it does not contain the neutral element, that is, the function $f(t) = 0$, for all t.

4.1.2 This subset V is a subspace of P_2:

- The neutral element $f(t) = 0$ (for all t) is in V.
- If f and g are in V (so that $f(2) = g(2) = 0$), then $(f+g)(2) = f(2) + g(2) = 0 + 0 = 0$, so that $f + g$ is in V.
- If f is in V (so that $f(2) = 0$), and k is any constant, then $(kf)(2) = kf(2) = 0$, so that kf is in V.

A polynomial $f(t) = a + bt + ct^2$ is in V if $f(2) = a + 2b + 4c = 0$, or $a = -2b - 4c$. The general element of V is of the form $f(t) = (-2b - 4c) + bt + ct^2 = b(t-2) + c(t^2 - 4)$, so that $t - 2$, $t^2 - 4$ is a basis of V.

4.1.3 This subset V is a subspace of P_2:

- The neutral element $f(t) = 0$ (for all t) is in V since $f'(1) = f(2) = 0$.
- If f and g are in V (so that $f'(1) = f(2)$ and $g'(1) = g(2)$), then

  $(f+g)'(1) = (f' + g')(1) = f'(1) + g'(1) = f(2) + g(2) = (f+g)(2)$, so that $f + g$ is in V.
- If f is in V (so that $f'(1) = f(2)$) and k is any constant, then $(kf)'(1) = (kf')(1) = kf'(1) = kf(2) = (kf)(2)$, so that kf is in V.

If $f(t) = a + bt + ct^2$ then $f'(t) = b + 2ct$, and f is in V if

$f'(1) = b + 2c = a + 2b + 4c = f(2)$, or $a + b + 2c = 0$. The general element of V is of the form $f(t) = (-b - 2c) + bt + ct^2 = b(t-1) + c(t^2 - 2)$, so that $t - 1, t^2 - 2$ is a basis of V.

4.1.4 This subset V is a subspace of P_2:

- The neutral element $f(t) = 0$ (for all t) is in V since $\int_0^1 0\, dt = 0$.

- If f and g are in V $\left(\text{so that } \int_0^1 f = \int_0^1 g = 0\right)$ then $\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0$, so that $f+g$ is in V.
- If f is in V $\left(\text{so that } \int_0^1 f = 0\right)$ and k is any constant, then $\int_0^1 kf = k\int_0^1 f = 0$, so that kf is in V.

If $f(t) = a + bt + ct^2$ then $\int_0^1 f(t)dt = \left[at + \frac{b}{2}t^2 + \frac{c}{3}t^3\right]_0^1 = a + \frac{b}{2} + \frac{c}{3} = 0$ if $a = -\frac{b}{2} - \frac{c}{3}$.

The general element of V is $f(t) = \left(-\frac{b}{2} - \frac{c}{3}\right) + bt + ct^2 = b\left(t - \frac{1}{2}\right) + c\left(t^2 - \frac{1}{3}\right)$, so that $t - \frac{1}{2},\ t^2 - \frac{1}{3}$ is a basis of V.

4.1.5 If $p(t) = a + bt + ct^2$ then $p(-t) = a - bt + ct^2$ and $-p(-t) = -a + bt - ct^2$.

Comparing coefficients we see that $p(t) = -p(-t)$ for all t if (and only if) $a = c = 0$.

The general element of the subset is of the form $p(t) = bt$.

These polynomials form a subspace of P_2, with basis t.

4.1.14 Yes

- $(0, 0, 0, \ldots, 0, \ldots)$ converges to 0.
- If $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0$.
- If $\lim_{n\to\infty} x_n = 0$ and k is any constant, then $\lim_{n\to\infty}(kx_n) = k\lim_{n\to\infty} x_n = 0$.

4.1.34 Let $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, meaning

$\begin{bmatrix} 3a+2c & 3b+2d \\ 4a+5c & 4b+5d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So $3a + 2c = a$, $3b + 2d = b$, $4a + 5c = c$ and $4b + 5d = d$. These imply that $a = -c$ and $b = -d$.

So the general element is $\begin{bmatrix} a & b \\ -a & -b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. Thus $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ is a basis, and the dimension is 2.

4.1.44 Let V be the space of all matrices S such that $AS = S\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$. Let's denote the column vectors of S by $\vec{u}, \vec{v}$ and $\vec{w}$. The condition $AS = S\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ means that $A\vec{u} = \vec{u}, A\vec{v} = \vec{v}$ and $A\vec{w} = \vec{0}$. This in turn means that the vectors $\vec{u}$ and $\vec{v}$ have to be on the plane V, while $\vec{w}$ is perpendicular to V. If we choose a basis $\vec{v}_1, \vec{v}_2$ of V and a nonzero vector $\vec{v}_3$ perpendicular to V, then we can write $\vec{u} = a\vec{v}_1 + b\vec{v}_2, \vec{v} = c\vec{v}_1 + d\vec{v}_2, \vec{w} = e\vec{v}_3$, and

$$S = [\vec{u}\ \vec{v}\ \vec{w}] = [a\vec{v}_1 + b\vec{v}_2\ c\vec{v}_1 + d\vec{v}_2\ e\vec{v}_3] = a[\vec{v}_1\ \vec{0}\ \vec{0}] + b[\vec{v}_2\ \vec{0}\ \vec{0}] + c[\vec{0}\ \vec{v}_1\ \vec{0}] + d[\vec{0}\ \vec{v}_2\ \vec{0}] + e[\vec{0}\ \vec{0}\vec{v}_3].$$

Thus $\dim(V) = 5$; the five matrices in the linear combination above form a basis of V.

4.1.52 We have to find constants a and b such that the functions e^{-x} and e^{-5x} are solutions of the differential equation $f''(x) + af'(x) + bf(x) = 0$. Thus it is required that $e^{-x} - ae^{-x} + be^{-x} = 0$, or $1 - a + b = 0$, and also that $25 - 5a + b = 0$. The solution of this system of two equations in two unknowns is $a = 6, b = 5$, so that the desired differential equation is $f''(x) + 6f'(x) + 5f(x) = 0$.

4.1.42 Let B be a matrix such that $\dim(\ker(B)) = k$. Then, it is required that the columns of A contain only vectors in the kernel of B. Thus, each column of A can be written as: $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$, where the vectors $\vec{v}_i$ form as basis of the kernel of B. Thus, each of the n columns in A involves k arbitrary constants, and matrix A involves nk arbitrary constants overall. The space of matrices A has dimension nk, where k is an integer in the range $[0, n]$.

4.1.56 Argue indirectly and assume that the space V of infinite sequences is finite-dimensional, with $\dim(V) = n$. According to the solution to Exercise 57, there can be at most n linearly independent elements in V. But here is our contradiction: It is easy to give $n + 1$ linearly independent infinite sequences, namely,

$(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots, (0, 0, 0, \ldots, 0, 1, 0, \ldots)$; in the last sequence the 1 is in the $(n + 1)$th place.

Ch 4.TF.**18** F; For any matrix A, the space of matrices commuting with A is at least two-dimensional. Indeed, if A is a scalar multiple of I_2, then A commutes with all 2×2 matrices, and if A fails to be a scalar multiple of I_2, then A commutes with the linearly independent matrices A and I_2.

Ch 4.TF.**30** T; Let our basis be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Each matrix here is invertible, and also clearly none are redundant.