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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

1)	T	F	If x^* is the least squares solution of $Ax = b$, then $\|\|b\|\|^2 = \|\|Ax^*\|\|^2 + \|\|b - Ax^*\|\|^2$.
2)	T	F	Similar matrices have the same determinant.
3)	T	F	If λ is an eigenvalue of A, then λ^3 is an eigenvalue of A^3.
4)	T	F	A shear in the plane is not diagonalizable.
5)	T	F	If A is invertible, then A and A^{-1} have the same eigenvectors.
6)	T	F	If A is a 3×3 matrix for which every entry is 1, then $\det(A) = 1$.
7)	T	F	If $\vec{v}$ is an eigenvector of A and of B and A is invertible, then $\vec{v}$ is an eigenvector of $3A^{-1} + 2B$.
8)	T	F	$\det(-A) = \det(A)$ for every 5×5 matrix A.
9)	T	F	If $\vec{v}$ is an eigenvector of A and an eigenvector of B and A is invertible, then $\vec{v}$ is an eigenvector of $A^{-3}B^2$.
10)	T	F	If a 11×11 matrix has the eigenvalues $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, then A is diagonalizable.
11)	T	F	For any $n \times n$ matrix, the matrix A has the same eigenvectors as A^T.
12)	T	F	If $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector of length 1, then $\vec{v}\vec{v}^T$ is a diagonalizable 3×3 matrix.
13)	T	F	The span of m orthonormal vectors is m-dimensional.
14)	T	F	A square matrix A can always be expressed as the sum of a symmetric matrix and a skew-symmetric matrix as follows $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.
15)	T	F	There exists an invertible $n \times n$ matrix A which satisfies $A^T A = A^2$, but A is not symmetric.
16)	T	F	If two $n \times n$ matrices A and B commute, then $(A^T)^2$ commutes with $(B^3)^T$.
17)	T	F	$-AA^T$ is skew-symmetric for every $n \times n$ matrix A.
18)	T	F	A matrix which is obtained from the identity matrix by an arbitrary number of switching of rows or columns is an orthogonal matrix.
19)	T	F	There exists a real 3×3 matrix A which satisfies $A^4 = -I_3$.
20)	T	F	Given 5 data points $(x_1, y_1), ..., (x_5, y_5)$, then a best fit with a polynomial $a + bt + ct^2 + dt^3 + et^4 + ft^5$ is possible in a unique way.

Problem 2) (10 points)

Check the boxes of all matrices which have zero determinants. You don't have to give justifications.

a) ☐ $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

b) ☐ $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$

c) ☐ $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

d) ☐ $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}$

e) ☐ $A = \begin{bmatrix} 10^{100} & 1 & 1 & 1 \\ 1 & 10^{100} & 1 & 1 \\ 1 & 1 & 10^{100} & 0 \\ 1 & 0 & 1 & 10^{100} \end{bmatrix}$

f) ☐ $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

g) ☐ $A = \begin{bmatrix} 11 & 10 & 8 & 5 \\ 9 & 7 & 4 & 0 \\ 6 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

h) ☐ $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

i) ☐ $A = \begin{bmatrix} 1 & 1 & 7 & 0 \\ 1 & 6 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

j) ☐ $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$

Problem 3) (10 points)

a) Find all (possibly complex) eigenvalues and eigenvectors of the matrix $Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b) Verify that Q^T has the same eigenvectors as Q.

c) Find a diagonal matrix B which is similar to the symmetric matrix

$$A = Q + Q^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

State algebraic and geometric multiplicities of the eigenvalues.

Problem 4) (10 points)

a) Let A be a $n \times n$ matrix such that $A^2 = 2A - I$. What are the possible eigenvalues of A?

b) Let A be a real $n \times n$ matrix such that $A^4 = -I_n$. Show that n must be even.

Problem 5) (10 points)

Find the function $f(t) = a + bt$ which best fits the data

$$\begin{aligned}
(x_1, y_1) &= (-1, 1) \\
(x_2, y_2) &= (0, 2) \\
(x_3, y_3) &= (1, 2) \\
(x_4, y_4) &= (3, 1) \\
(x_5, y_5) &= (3, 0)
\end{aligned}$$

Problem 6) (10 points)

Find the determinant of the matrix

$$\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
-1 & 0 & 3 & 4 & 5 & 6 & 7 & 8 \\
-1 & -2 & 0 & 4 & 5 & 6 & 7 & 8 \\
-1 & -2 & -3 & 0 & 5 & 6 & 7 & 8 \\
-1 & -2 & -3 & -4 & 0 & 6 & 7 & 8 \\
-1 & -2 & -3 & -4 & -5 & 0 & 7 & 8 \\
-1 & -2 & -3 & -4 & -5 & -6 & 0 & 8 \\
-1 & -2 & -3 & -4 & -5 & -6 & -7 & 0
\end{bmatrix}.$$

Show your work carefully.

Problem 7) (10 points)

A discrete dynamical system is given by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x - 2y \\ -x + 3y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} .$$

Find a closed formula for $T^{100}(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$.

Problem 8) (10 points)

a) Show that for an arbitrary matrix A for which $A^T A$ is invertible, the least squares solution of $A\vec{x} = \vec{b}$ simplifies to

$$\vec{x} = R^{-1} Q^T \vec{b} ,$$

if $A = QR$ is the QR decomposition of A.

b) Find the least square solution in the case $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ using this formula.

Problem 9) (10 points)

a) (7 points) Find the QR decomposition of

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

then form the new matrix $T(A) = RQ$.

b) (3 points) Verify that for any invertible $n \times n$ matrix $A = QR$, the matrix $T(A) = RQ$ has the same eigenvalues as A.

c) Find the least square solution for the system $A\vec{x} = \vec{b}$ given by the equations

$$\begin{aligned} x + y &= 4 \\ y &= 2 \\ x &= -1 . \end{aligned}$$