3/2/2010 FIRST HOURLY **Math 21b, Spring 2010**

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-2, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F A reflection is its own inverse.

2) T F Every basis of R^n has the same number of vectors in it.

3) T F The rank of a 3×7 matrix is smaller or equal than 3.

4) T F If $\{v_1, v_2, v_3, v_4\}$ is a set of linearly independent vectors of a linear space V, then $\dim(V) \geq 4$.

5) T F If A is a 4×5 matrix, then the dimension of $\ker(A)$ is at least one.

6) T F The sum $A + B$ of 2 invertible matrices A, B is invertible.

7) T F If A, B, and C are $n \times n$ matrices, then the property $A(B+C) = AB + AC$ holds.

8) T F If $A\vec{x} = \vec{0}$ has no nonzero solutions, where A is a 4×3 matrix, then $\text{rank}(A) = 3$.

9) T F If $\vec{b}$ is in $\text{im}(A)$, then $A\vec{x} = \vec{b}$ exactly one solution.

10) T F If A is a nonzero column vector with 2 components and B is the same vector written as a row vector then AB is invertible.

11) T F If $\vec{v}$ is a redundant column vector of A, then $\vec{v}$ is in the kernel of A.

12) T F If $AB = I_n$ for an $n \times m$ matrix A and B is a $m \times n$ matrix, then $BA = I_m$.

13) T F The product of 2 invertible 3×3 matrices is invertible.

14) T F Suppose A and B are $n \times n$ matrices. If A is invertible and B is not, then AB is not invertible.

15) T F If a 2×2 matrix different from the identity is its own inverse then it is a reflection at a line.

16) T F If a linear transformation from R^n to R^n has no nontrivial kernel, then it is invertible.

17) T F The set of quadratic polynomials $ax^2 + bx + c$ has a basis consisting of 2 vectors.

18) T F The set of vectors (x, y) in R^2 such that $xy > 0$ is a linear subspace of R^2.

19) T F The space of functions $ax + bx^2 + ce^x$ with real a, b, c is a linear space.

20) T F The solution set to the system of equations $x+y = 1, 2x+2y = 2$ is a linear space.

Total

Problem 2) (10 points) No justifications are needed.

a) (3 points) One of the following matrices can be composed with a dilation to become an orthogonal projection onto a line. Which one?

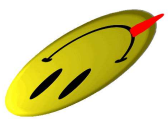
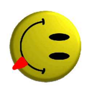
$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

b) (4 points)
The **smiley face** visible to the right is transformed with various linear transformations represented by matrices $A - F$. Find out which matrix does which transformation:

$$\text{A=} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{B=} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{C=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\text{D=} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad \text{E=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{F=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}/2$$

A-F	image	A-F	image	A-F	image
					
					

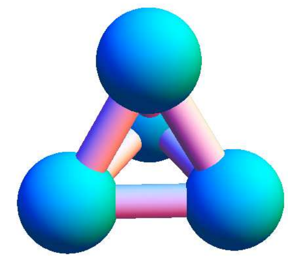
c) (3 points) Which of the following sets are linear spaces? (The variables a, b, c, d are constants.)

The space of all ...	Check
functions $f(x) = a + \frac{b}{1+x^2}$	
diagonal matrices	
vertical shear 2×2 matrices	

The space of all ...	Check
invertible matrices S on R^3	
points (x, y) with $3x^2 + 2y = 0$	
functions $f(x, y) = cx^2 + dy$	

Problem 3) (10 points)

A **tetrahedral molecule** has four corners with charges x, y, z, w. We know the sum of the charges of three of the four faces and want to determine the individual charges. Find all possible charge distributions.



$$\left| \begin{array}{ccccccccc} x & + & y & + & z & & & = & 12 \\ x & + & y & & & + & w & = & 9 \\ x & + & & & z & + & w & = & 8 \end{array} \right|$$

Problem 4) (10 points)

Find a basis of the image and a basis for the kernel of the following **Pascal triangle matrix**:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \end{bmatrix}.$$

Problem 5) (10 points)

a) (5 points) Invert the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

using row reduction.

b) (5 points) Find a basis for the space of vectors perpendicular to the image of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

Problem 6) (10 points)

Let T be the linear transformation from R^3 to R^3 which is an orthogonal projection onto the plane

$$2x - y + 5z = 0 .$$

In the standard basis, we have $T(x) = Ax$.

a) (2 points) What is the dimension of $\ker(A)$?
b) (4 points) Find a basis $\mathcal{B}$ so that T is described by a diagonal matrix B.
c) (2 points) Find that diagonal matrix B.
d) (2 points) Which of the following relationships between A and B hold? Check all which apply:

Property	Check if true
A is similar to B	
$AB = BA$	
$SA = BS$ for some matrix S	
$A = MBM^{-1}$ for some matrix M	

Problem 7) (10 points)

Let $S(\vec{x}) = A\vec{x}$ be the transformation given by

$$A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} /2$$

and $T(\vec{x}) = B\vec{x}$ be the dilation (=scaling) transformation by the dilation factor $\frac{1}{2}$. Finally, let $R = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ be a horizontal shear.

a) (4 points) If $M(\vec{x}) = T(S(\vec{x}))$, find the matrix which gives the transformation

$$M^{13} = M(M(M(M(M(M(M(M(M(M(M(M(M(\vec{x}))))))))))))) .$$

b) (4 points) Let $U(\vec{x}) = R\vec{x}$ where If $L(\vec{x}) = U(S(\vec{x}))$, find the matrix which gives the transformation L.
c) (2 points) Is $AR = RA$?

Problem 8) (10 points)

a) (5 points) What are the coordinates of the vector $\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$ in the basis

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \} ?$$

b) (5 points) A transformation T is described in the standard basis by

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} .$$

What is the matrix B of the transformation T in the basis $\mathcal{B}$?

Problem 9) (10 points)

In this problem, we consider the following four matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}, C = \begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} .$$

a) (5 points) For each of the matrices A, B, C, find the row reduced echelon form and the rank. Decide further whether the matrix is invertible and whether the matrix is similar to the matrix D:

matrix M	rref(M)=	rank(M) =	invertible?	similar to D?
A	$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$			
B	$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$			
C	$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$			

b) (5 points) One of the matrices A, B, C is a reflection-dilation (a reflection composed with a dilation), one is a projection dilation (a projection composed with a dilation) and one is a rotation dilation (a rotation composed with a dilation). Identify which is which. Find the dilation factor. For the reflection dilation matrix, determine the line about which the reflection takes place. For the projection dilation matrix, find the line onto which it projects. For the rotation dilation matrix determine the angle of rotation.

Type	Matrix (enter A,B,C)	dilation factor	angle or line
rotation dilation			angle=
projection dilation			projection line =
reflection dilation			reflection line =