FINAL EXAM, May 8, 2010 **Math 21b, Spring 10**

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- Please write your name above and check your section to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details and justifications. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) True or False? No justifications are needed.

1) T F The inverse of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is equal to $A/2$.

2) T F The sum of two eigenvectors v_1, v_2 of a matrix A is an eigenvector of A.

3) T F If S is the matrix which contains an eigenbasis of a $n \times n$ matrix A as columns, then $AS = DS$, where D is a diagonal matrix.

4) T F A 2×2 matrix A is diagonalizable if $A + A^T$ is diagonalizable.

5) T F For the function $f(x) = |\sin(3x)| + |x|$, all Fourier coefficients of $\cos(nx)$ are zero.

6) T F All continuous functions satisfying $2\pi f(x) = f(\pi(x+1)) + f(\pi(x-1))$ form a linear space.

7) T F If the sum of all algebraic multiplicities of a 3×3 matrix A is equal to 3, then the matrix is diagonalizable.

8) T F Every 2×2 matrix can be written as a product of a rotation, a diagonal matrix and a shear.

9) T F A discrete dynamical system for which all the eigenvalues are negative is asymptotically stable.

10) T F A system of linear equations $A\vec{x} = \vec{b}$ which is consistent has exactly one solution.

11) T F A rotation composed with a reflection is an orthogonal transformation.

12) T F If $f(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2 t} \sin(nx)$ solves the heat equation, then $\langle f(x,0), \sin(3x) \rangle = 1/27$, where $\langle f, g \rangle = \frac{2}{\pi} \int_0^{\pi} f(x)g(x)\, dx$.

13) T F For a 5×6 matrix A the equation $Ax = b$ has either zero or infinitely many solutions.

14) T F If the Jacobian matrix at a equilibrium point (a,b) of a nonlinear system $x' = f(x,y)$, $y' = g(x,y)$ is orthogonal, then the point (a,b) is asymptotically stable.

15) T F If a matrix A is diagonalizable, and $A = QR$ is the QR decomposition, then R is diagonalizable.

16) T F $||3\sin(5x) + 4\cos(20x)|| = 25$, where the length $||f||$ of a function f is taken with respect to the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx$.

17) T F If the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ is asymptotically stable, then also $\vec{x}(t+1) = (A^2 + A)\vec{x}(t)$ is asymptotically stable.

18) T F If $dx/dt = Ax$ is a differential equation where A is a 2×2 matrix with trace 10, then $(0,0)$ can not be asymptotically stable.

19) T F If A is a symmetric 4×4 matrix satisfying $A^2 = A$, then the algebraic multiplicity is at least 2 for some eigenvalue of A.

20) T F If A is a reflection at the line $3x = 4y$ and B is a projection onto $x = y$ then $A + B$ is diagonalizable.

Problem 2) (10 points) no justifications needed

a) (6 points) In this problem, we deal with geometric linear transformations in the plane. Match the transformations with the trace and determinant values to the left. Each transformation matches exactly one of the cases.

trace	det	enter A-F
1	0	
0	-1	
0	1	
2	1	
4	4	
-2	1	

label	transformation
A	rotation by 90 degrees
B	projection onto a line
C	dilation by 2
D	reflection at a line
E	shear
F	rotation by 180 degrees

b) (4 points) Match the initial value problem with their solutions. Each function matches exactly one of the differential equations.

enter A-D	initial value problem
	$f''(t) = 4, f(0) = 4, f'(0) = 0$
	$f''(t) + 4f(t) = 4, f(0) = 2, f'(0) = 2$
	$f''(t) - 4f(t) = -4, f(0) = 3, f'(0) = 0$
	$f'(t) - 4f(t) = 4, f(0) = 4$

label	solution
A)	$f(t) = e^{2t} + e^{-2t} + 1$
B)	$f(t) = 1 + \cos(2t) + \sin(2t)$
C)	$f(t) = 5e^{4t} - 1$
D)	$f(t) = 4 + 2t^2$

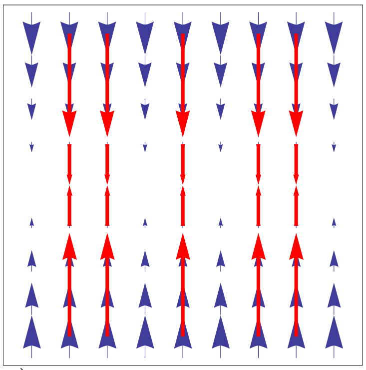
Problem 3) (10 points) no justifications needed

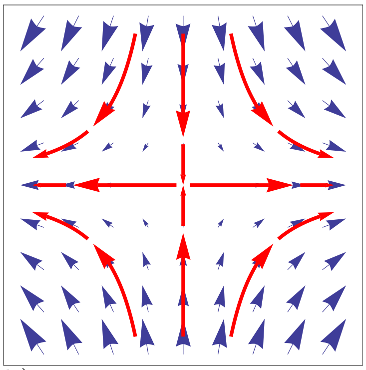
a) (5 points) Suppose you know that the eigenvalues of a 2×2 matrix A are $\lambda_1 = 2$ with eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 3$ with eigenvector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Check the boxes which apply.

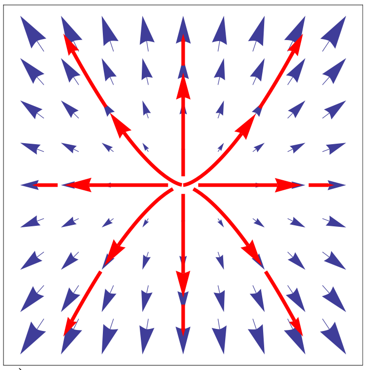
If $B =$	2 and 3 are eigenvalues of B	v_1 and v_2 are both eigenvectors of B
A^T		
$A^2 - A$		
A^{-1}		

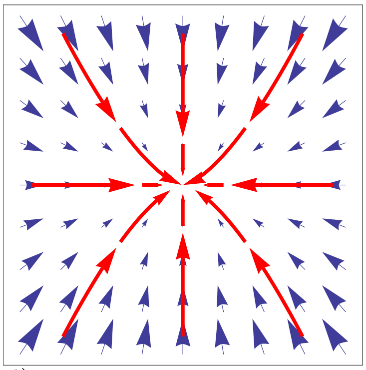
b) (5 points) Associate the dynamical system with the phase portraits.

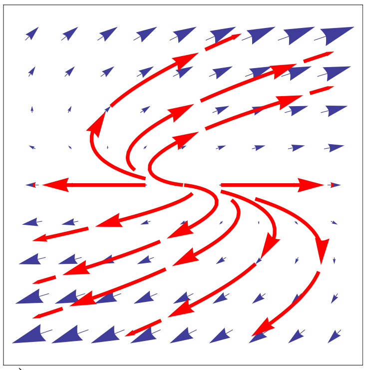
matrix	portrait a)-f) for $\frac{d}{dt}x = Ax$
$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$	
$A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$	
$A = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$	

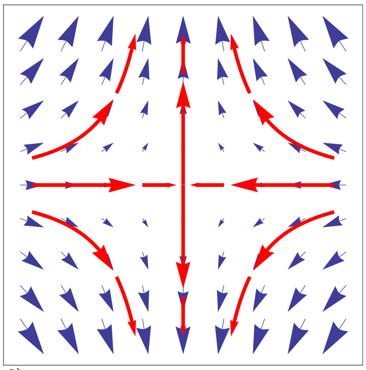

a)


b)


c)


d)


e)


f)

Problem 4) (10 points)

Find all the solutions of the following system of linear equations. Use basic row reduction steps.

$$\left| \begin{array}{cccccccccl} x & & & & & & & + & v & = 10 \\ & & y & + & z & & & & & = 6 \\ x & & & & & + & u & + & v & = 6 \end{array} \right|$$

Problem 5) (10 points)

Find the best function

$$f(x,y) = a\sin(x) + b\cos(y) = z$$

which fits the data points $(0,0,1), (\pi/2,0,2), (\pi/2,\pi/2,4)$ using the least square method.

Problem 6) (10 points)

a) (4 points) Find all the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} .$$

b) (1 point) is the continuous dynamical system $d/dt\vec{x}(t) = B\vec{x}(t)$ defined by the matrix B in a) asymptotically stable?

c) (4 points) A regular 3×3 transition matrix A defines a discrete dynamical system $\vec{x}(n+1) = A\vec{x}(n)$: which has the property that $A^n v$ converges to a multiple of an eigenvector belonging to the largest eigenvalue of

$$A = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 2/3 & 1/3 \\ 2/3 & 0 & 1/3 \end{bmatrix} .$$

Find this eigenvector of A.

d) (1 point) Is the discrete dynamical system $\vec{x}(n+1) = A\vec{x}(n)$ from part c) asymptotically stable?

Problem 7) (10 points)

Assume the matrix A has an eigenbasis

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$$

which belongs to the eigenvalues $1, 2$ and 3 given in the same order. Find the matrix A.

Problem 8) (10 points)

Let's have a look at the **multiplication table** we learned as kids in first grade:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{bmatrix} .$$

Find all eigenvalues and eigenvectors of this symmetric matrix.

Problem 9) (10 points)

a) (2 points) Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} .$$

b) (4 points) When studying the "quintic threefold" in string theory, the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

appears. Find the determinant of A.

c) (4 points) **Werner Heisenberg** formulated quantum mechanics using matrices. The truncated $n \times n$ version of the position operator matrix is called A_n. We have for example

$$A_8 = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & \sqrt{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7} & 0 \end{bmatrix}$$

Find $\det(A_2)$. Explain, why $\det(A_4) = (-3)\det(A_2)$ and $\det(A_6) = (-5)\det(A_4)$. Do the same for $\det(A_8) = (-7)\det(A_6)$. Use this to compute $\det(A_8)$.

Problem 10) (10 points)

Find the general solutions for the following differential equations:

a) (5 points)

$$f''(t) + f(t) = 3\sin(2t) + 1, f(0) = 3, f'(0) = 0$$

b) (5 points)

$$f''(t) - 2f'(t) + f(t) = t^2, f(0) = 2, f'(0) = -4$$

Problem 11) (10 points)

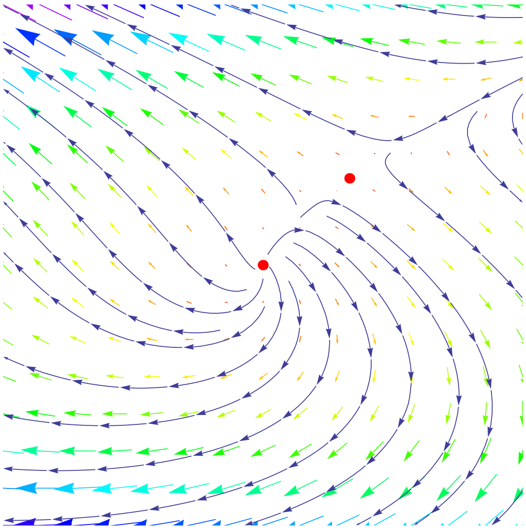
Analyze the solutions $(x(t), y(t))$ for the following nonlinear dynamical system

$$\begin{aligned} \frac{d}{dt}x &= x - y^2 \\ \frac{d}{dt}y &= y - x \end{aligned}$$

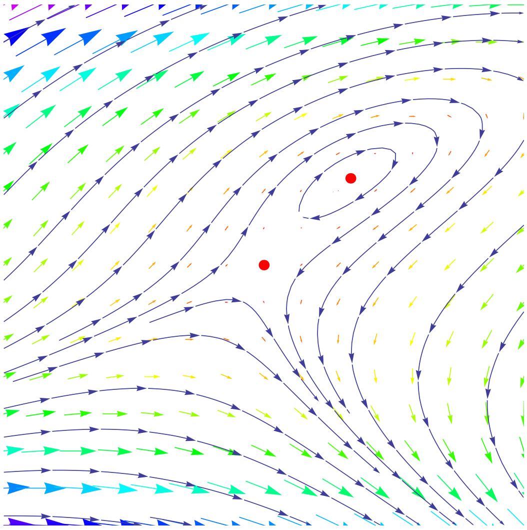
a) (3 points) Find the equations of the null-clines and find all the equilibrium points.

b) (4 points) Analyze the stability of all the equilibrium points.

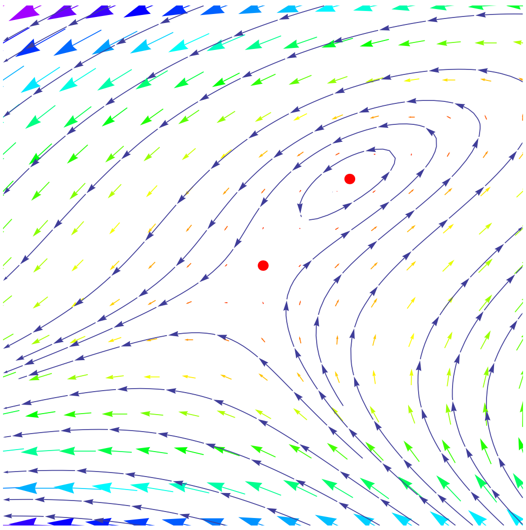
c) (3 points) Which of the phase portraits A,B,C,D below belongs to the above system?



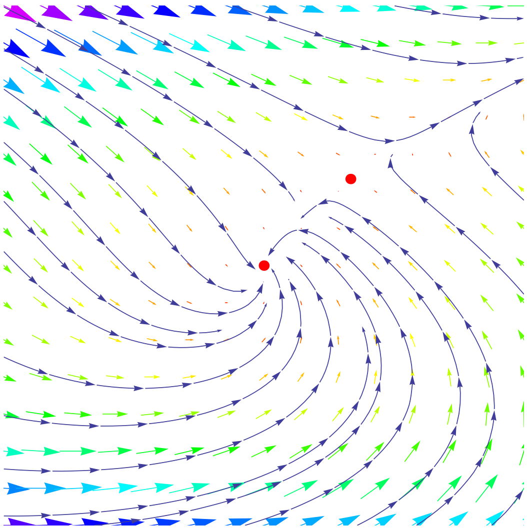
A



B



C



D

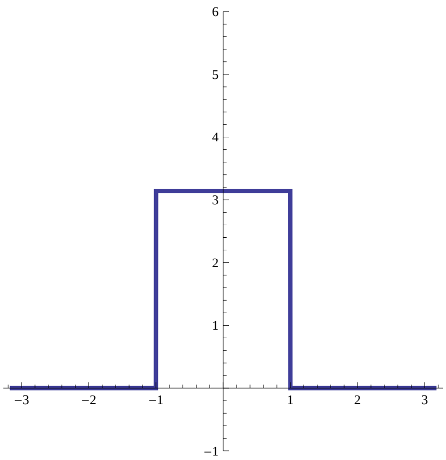
Problem 12) (10 points)

a) (4 points) Find the Fourier series of the function

$$f(x) = \begin{cases} \pi & , \quad -1 \leq x \leq 1 \\ 0 & , \quad \text{else} \end{cases} .$$

The graph of the function is visible to the right.

Note: Leave terms like $\cos(n), \sin(n)$ as they are.



b) (3 points) Use Parseval's theorem to find the value of the sum

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2} .$$

c) (3 points) We add a parameter $0 \leq a \leq \pi$ in the above computation:

$$f(x) = \begin{cases} \pi & , \quad -a \leq x \leq a \\ 0 & , \quad \text{else} \end{cases} .$$

What is

$$g(a) = \sum_{n=1}^{\infty} \frac{\sin^2(na)}{n^2} \ ?$$

Problem 13) (10 points)

A **forest fire** temperature is given by a function $f(x, y, t)$ for (x, y) in a square $[0, \pi] \times [0, \pi]$. Since the burning intensity becomes bigger with more heat and additionally, the fire diffuses, the temperature $f(x, y, t)$ at the point (x, y) and time t satisfies the partial differential equation

$$f_t = T(f) = f_{xx} + f_{yy} + 11 \cdot f .$$

a) (2 points) The operator $T = D_x^2 + D_y^2 + 11$ has eigenfunctions $\sin(nx)\sin(my)$. What are the eigenvalues?

b) (4 points) Assume the fire has initially the temperature

$$f(x, y) = \sin(4x)\sin(7y) .$$

What is the temperature at a later time t? Does it die out?

c) (4 points) Now assume the fire has initially the temperature

$$f(x, y) = \sin(3x)\sin(y) .$$

What happens now?