Solutions

2.4.**14** Use Theorem 2.4.5; the inverse is $\begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

2.4.**28** We are asked to find the inverse of the matrix $A = \begin{bmatrix} 22 & 13 & 8 & 3 \\ -16 & -3 & -2 & -2 \\ 8 & 9 & 7 & 2 \\ 5 & 4 & 3 & 1 \end{bmatrix}$.

We find that $A^{-1} = \begin{bmatrix} 1 & -2 & 9 & -25 \\ -2 & 5 & -22 & 60 \\ 4 & -9 & 41 & -112 \\ -9 & 17 & 80 & 222 \end{bmatrix}$.

T^{-1} is the transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ with matrix A^{-1}.

2.4.**32** Use Theorem 2.4.9.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix such that $ad - bc = 1$ and $A^{-1} = A$, then

$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so that $b = 0,\ c = 0$, and $a = d$.

The condition $ad - bc = a^2 = 1$ now implies that $a = d = 1$ or $a = d = -1$.

This leaves only two matrices A, namely, I_2 and $-I_2$. Check that these two matrices do indeed satisfy the given requirements.

2.4.**44** a $\text{rref}(M_4) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so that $\text{rank}(M_4) = 2$.

b To simplify the notation, we introduce the row vectors $\vec{v} = [1\ 1\ \ldots\ 1]$ and $\vec{w} = [0\ n\ 2n\ \ldots\ (n-1)n]$ with n components.

Then we can write M_n in terms of its rows as $M_n = \begin{bmatrix} \vec{v} + \vec{w} \\ 2\vec{v} + \vec{w} \\ \cdots \\ n\vec{v} + \vec{w} \end{bmatrix} \begin{matrix} \\ -2(I) \\ \cdots \\ -n(I) \end{matrix}$.

Applying the Gauss-Jordan algorithm to the first column we get $\begin{bmatrix} \vec{v}+\vec{w} \\ -\vec{w} \\ -2\vec{w} \\ \dots \\ -(n-1)\vec{w} \end{bmatrix}$.

All the rows below the second are scalar multiples of the second; therefore, $\text{rank}(M_n) = 2$.

c By part (b), the matrix M_n is invertible only if $n = 1$ or $n = 2$.

2.4.**67** Not necessarily true; $(A+B)^2 = (A+B)(A+B) = A^2+AB+BA+B^2 \neq A^2+2AB+B^2$ if $AB \neq BA$.

2.4.**68** True; apply Theorem 2.4.7 to $B = A$.

2.4.**69** Not necessarily true; consider the case $A = I_n$ and $B = -I_n$.

2.4.**70** Not necessarily true; $(A-B)(A+B) = A^2 + AB - BA - B^2 \neq A^2 - B^2$ if $AB \neq BA$.

2.4.**71** True; $ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$.

2.4.**72** Not necessarily true; the equation $ABA^{-1} = B$ is equivalent to $AB = BA$ (multiply by A from the right), which is not true in general.

2.4.**92** We try to find matrices $L = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ and $U = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae \\ bd & be+cf \end{bmatrix}.$$

Note that the equations $ad = 0,\ ae = 1$, and $bd = 1$ cannot be solved simultaneously: If $ad = 0$ then a or d is 0 so that ae or bd is zero.

Therefore, the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does not have an LU factorization.