

## Solutions

5.5.10 A function  $g(t) = a + bt + ct^2$  is orthogonal to  $f(t) = t$  if

$$\langle f, g \rangle = \int_{-1}^1 (at + bt^2 + ct^3) dt = \left[ \frac{a}{2}t^2 + \frac{b}{3}t^3 + \frac{c}{4}t^4 \right]_{-1}^1 = \frac{2}{3}b = 0, \text{ that is, if } b = 0.$$

Thus, the functions 1 and  $t^2$  form a basis of the space of all functions in  $P_2$  orthogonal to  $f(t) = t$ . To find an *orthonormal* basis  $g_1(t), g_2(t)$ , we apply Gram-Schmidt. Now

$$\|1\| = \frac{1}{2} \int_{-1}^1 1 dt = 1, \text{ so that we can let } g_1(t) = 1. \text{ Then } g_2(t) = \frac{t^2 - \langle 1, t^2 \rangle 1}{\|t^2 - \langle 1, t^2 \rangle 1\|} = \frac{t^2 - \frac{1}{3}}{\|t^2 - \frac{1}{3}\|} = \frac{\sqrt{5}}{2}(3t^2 - 1)$$

$$\text{Answer: } g_1(t) = 1, g_2(t) = \frac{\sqrt{5}}{2}(3t^2 - 1)$$

5.5.11  $\langle f, g \rangle = \langle \cos(t), \cos(t+\delta) \rangle = \langle \cos(t), \cos(t)\cos(\delta) - \sin(t)\sin(\delta) \rangle = \cos(\delta)\langle \cos(t), \cos(t) \rangle - \sin(\delta)\langle \cos(t), \sin(t) \rangle = \cos(\delta)$ , by Theorem 5.5.4.

Also,  $\langle g, g \rangle = 1$  (left to reader).

$$\text{Thus, } \angle(f, g) = \arccos\left(\frac{\langle f, g \rangle}{\|f\|\|g\|}\right) = \arccos(\cos \delta) = \delta.$$

5.5.16 a We start with the standard basis 1,  $t$  and use the Gram-Schmidt process to construct an *orthonormal* basis  $g_1(t), g_2(t)$ .

$$\|1\| = \sqrt{\int_0^1 dt} = 1, \text{ so that we can let } g_1(t) = 1. \text{ Then } g_2(t) = \frac{t - \langle 1, t \rangle 1}{\|t - \langle 1, t \rangle 1\|} = \frac{t - \frac{1}{2}}{\|t - \frac{1}{2}\|} = \sqrt{3}(2t - 1).$$

*Summary:*  $g_1(t) = 1$  and  $g_2(t) = \sqrt{3}(2t - 1)$  is an orthonormal basis.

b We are looking for  $\text{proj}_{P_1}(t^2) = \langle g_1(t), t^2 \rangle g_1(t) + \langle g_2(t), t^2 \rangle g_2(t)$ , by Theorem 5.5.3.

We find that  $\langle g_1(t), t^2 \rangle = \int_0^1 t^2 dt = \frac{1}{3}$  and  $\langle g_2(t), t^2 \rangle = \sqrt{3} \int_0^1 (2t^3 - t^2) dt = \frac{\sqrt{3}}{6}$ , so that  $\text{proj}_{P_1} t^2 = \frac{1}{3} + \frac{1}{2}(2t - 1) = t - \frac{1}{6}$ . See Figure 5.21.

5.5.24 a  $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle = 0 + 8 = 8$

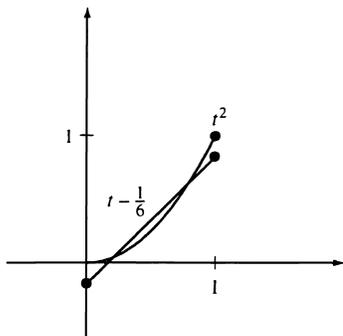


Figure 1: for Problem 5.5.16b.

$$\text{b } \|g + h\| = \sqrt{\langle g + h, g + h \rangle} = \sqrt{\langle g, g \rangle + 2\langle g, h \rangle + \langle h, h \rangle} = \sqrt{1 + 6 + 50} = \sqrt{57}$$

c Since  $\langle f, g \rangle = 0$ ,  $\|g\| = 1$ , and  $\|f\| = 2$ , we know that  $\frac{f}{2}, g$  is an orthonormal basis of  $\text{span}(f, g)$ .

$$\text{Now } \text{proj}_E h = \left\langle \frac{f}{2}, h \right\rangle \frac{f}{2} + \langle g, h \rangle g = \frac{1}{4} \langle f, h \rangle f + \langle g, h \rangle g = 2f + 3g.$$

d From part c we know that  $\frac{1}{2}f, g$  are orthonormal, so we apply Theorem 5.2.1 to obtain the third polynomial in an orthonormal basis of  $\text{span}(f, g, h)$ :

$$\frac{h - \text{proj}_E h}{\|h - \text{proj}_E h\|} = \frac{h - 2f - 3g}{\|h - 2f - 3g\|} = \frac{h - 2f - 3g}{5} = -\frac{2}{5}f - \frac{3}{5}g + \frac{1}{5}h$$

$$\text{Orthonormal basis: } \frac{1}{2}f, g, -\frac{2}{5}f - \frac{3}{5}g + \frac{1}{5}h$$

5.5.32 a  $\langle t^n, t^m \rangle = \frac{1}{2} \int_{-1}^1 t^{n+m} dt = \frac{1}{2} \frac{t^{n+m+1}}{n+m+1} \Big|_{-1}^1$  If  $n + m$  is even, this is  $1/(n + m + 1)$ . If  $n + m$  is odd, this is zero.

$$\text{b } \|t^n\| = \left( \frac{1}{2} \int_{-1}^1 t^{2n} dt \right)^{1/2} = \left( t^{2n+1} \Big|_{-1}^1 / 2 \right)^{1/2} = (4n + 2)^{-1/2} \text{ by part a).}$$

c  $g_0(t) = 1$ . We obtain  $g_1(t)$  by normalizing  $f_1(t) = t - \langle t, 1 \rangle t = t$ . Because  $\|t\| = 1/\sqrt{3}$  we have  $g_1(t) = \sqrt{3}t$ . We get  $g_2(t)$  by normalizing  $f_2(t) = t^2 - 1 \langle t^2, 1 \rangle - \sqrt{3}t \langle t^2, \sqrt{3}t \rangle = t^2 - 1/3$  which has norm  $\|f_2\| = 2/(3\sqrt{5})$ . Therefore  $g_2(t) = 3\sqrt{5}t^2/2 - \sqrt{5}/2 = \frac{\sqrt{5}}{2}(3t^2 - 1)$ . Finally, we get  $g_3(t)$  by normalizing

$$f_3(t) = t^3 - 1 \langle t^3, 1 \rangle - \sqrt{3}t \langle t^3, \sqrt{3}t \rangle - (3\sqrt{5}t^2/2 - \sqrt{5}/2) \langle t^3, 3\sqrt{5}t^2/2 - \sqrt{5}/2 \rangle$$

which simplifies to  $t^3 - 3t/5$ . It has norm  $\|f_3\| = 2/(5\sqrt{7})$  so that  $g_3(t) = (5t^3 - 3t)\sqrt{7}/2$ .

d  $g_0(t)/g_0(1) = 1$ ,  $g_1(t)/g_1(1) = t$ ,  $g_2(t)/g_2(1) = (3t^2 - 1)/2$ , and  $g_3(t)/g_3(1) = (5t^3 - 3t)/2$ .

e The polynomial which best approximates  $f$  is  $\langle f, g_0 \rangle g_0 + \langle f, g_1 \rangle g_1 + \langle f, g_2 \rangle g_2 + \langle f, g_3 \rangle g_3 = 15(\pi - 3)t^2/4 + (3\pi/2 - 15/4)$ . The computation can be simplified by noting that  $f$  is an even function, so that  $\langle f, g_1 \rangle = \langle f, g_3 \rangle = 0$ .

