DIFFERENTIAL OPERATORS Math 21b, 2010

Problem 1) Check in each case the three properties. a) $T(f)(x) = x^2 f(x-4)$ is linear.
b) $T(f)(x) = f'(x)^2$ is not linear. The property (ii) fails.
c) $T = D^2 + D + 1$ is linear.
d) $T(f)(x) = e^x \int_0^x e^{-t} f(t)\, dt$ is linear.

Problem 2) a) The kernel of $T = D + 3$ is $\{Ce^{-3x}\}$. The kernel of $D - 2$ is $\{Ce^{2x}\}$. The kernel of $(D+3)(D-2)$ contains these two kernels and is $\{Ae^{-3x} + Be^{2x}\}$. b) With $f(x) = e^{-x^2/2}$ we would get $-xf(x)$. So, $f' + xf(x) = 0$. But $f(x) = xe^{-x^2/2}$ is not in the kernel.

Problem 3) a) $e^{(\lambda/i)x}$ is an eigenfunction.
b) $[Q,P]f = QPf - PQf = ixf'(x) - id/dx(xf(x)) = -if(x)$. So $[Q,P] = -i$.

Problem 4) Define $f = e^{3x} \int_0^x e^{-3t} \sin(t)\, dt$. Differentiation gives, using the fundamental theorem of calculus $f' = 3f(x) + e^{3x}e^{-3x}\sin(x)$. Therefore $f' - 3f = \sin(x)$.

Problem 5) a) Differentiate $f(x) = e^{-x^2/2}$ twice with respect to x. This gives $x^2 f(x) + f(x)$. The eigenvalue is 1.
b) Differentiate $f(x) = xe^{-x^2/2}$ twice with respect to x. This gives $x^3 e^{-x^2/2} + 3f(x)$ The eigenvalue is 3.