Solutions

1.3.**10** $\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\1\end{bmatrix} = 1\cdot 1 + 2\cdot(-2) + 3\cdot 1 = 0$

1.3.**12** $[1\ 2\ 3\ 4] \cdot \begin{bmatrix}5\\6\\7\\8\end{bmatrix} = 1\cdot 5 + 2\cdot 6 + 3\cdot 7 + 4\cdot 8 = 70$

1.3.**26** From Example 3d we know that rank$(A) = 3$, so that rref$(A) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$.

Since all variables are leading, the system $A\vec{x} = \vec{c}$ cannot have infinitely many solutions, but it could have a unique solution (for example, if $\vec{c} = \vec{b}$) or no solutions at all (compare with Example 3c).

1.3.**28** There must be a leading one in each column: rref $(A) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}$.

1.3.**36** By Exercise 35, the ith column of A is $A\vec{e}_i$, for $i = 1, 2, 3$. Therefore, $A = \begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$.

1.3.**38** We will illustrate our reasoning with an example. We generate the "random" 3×3 matrix

$$A = \begin{bmatrix}0.141&0.592&0.653\\0.589&0.793&0.238\\0.462&0.643&0.383\end{bmatrix}.$$

Since the entries of this matrix are chosen from a large pool of numbers (in our case 1000, from 0.000 to 0.999), it is unlikely that any of the entries will be zero (and even less likely that the whole first column will consist of zeros). This means that we will usually be able to apply Steps 2 and 3 of the Gauss-Jordan algorithm to turn the first column into $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; this is indeed possible in our example: $\begin{bmatrix} 0.141 & 0.592 & 0.653 \\ 0.589 & 0.793 & 0.238 \\ 0.462 & 0.643 & 0.383 \end{bmatrix} \longrightarrow$ $\begin{bmatrix} 1 & 4.199 & 4.631 \\ 0 & -1.680 & -2.490 \\ 0 & -1.297 & -1.757 \end{bmatrix}$.

Again, it is unlikely that any entries in the second column of the new matrix will be zero. Therefore, we can turn the second column into $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Likewise, we will be able to clear up the third column, so that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We summarize:

As we apply Gauss-Jordan elimination to a random matrix A (of any size), it is unlikely that we will ever encounter a zero on the diagonal. Therefore, rref(A) is likely to have all ones along the diagonal.

1.3.**46** Since a, d, and f are all nonzero, we can divide the first row by a, the second row by d, and the third row by f to obtain

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ 0 & 1 & \frac{e}{d} \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that the rank of the matrix is 3.

1.3.**56** We can use technology to determine that the system $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix} +$ $x_3 \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}$ is inconsistent; therefore, the vector $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$ fails to be a linear combination of the other four vectors.