

## Solutions

8.1.6  $\frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$  is an orthonormal eigenbasis.

8.1.10  $\lambda_1 = \lambda_2 = 0$  and  $\lambda_3 = 9$ .

$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$  is in  $E_0$  and  $\vec{v}_2 = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$  is in  $E_9$ .

Let  $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}$ ; then  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is an orthonormal eigenbasis.

$$S = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{3} & -\frac{4}{3\sqrt{5}} \\ 0 & \frac{2}{3} & -\frac{\sqrt{5}}{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8.1.24 Note that  $A$  is symmetric and orthogonal, so that the eigenvalues are 1 and  $-1$  (see Exercise 23).

$E_1 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$  and  $E_{-1} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right)$ , so that

$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$  is an orthonormal eigenbasis.

8.1.28 For  $\lambda \neq 0$

$$f_A(\lambda) = \det \left[ \begin{array}{cccc|c} -\lambda & & & 0 & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ 0 & & & -\lambda & 1 \\ 1 & 1 & \cdots & 1 & 1-\lambda \end{array} \right] = \frac{1}{\lambda} \det \left[ \begin{array}{cccc|c} -\lambda & & & 0 & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ & & & -\lambda & 1 \\ \lambda & \lambda & \cdots & \lambda & \lambda - \lambda^2 \end{array} \right]$$

$$= \frac{1}{\lambda} \det \left[ \begin{array}{cccc|c} -\lambda & & & & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ & & & -\lambda & 1 \\ 0 & 0 & \cdots & 0 & -\lambda^2 + \lambda + 12 \end{array} \right]$$

$$= -\lambda^{11}(\lambda^2 - \lambda - 12) = -\lambda^{11}(\lambda - 4)(\lambda + 3)$$

Eigenvalues are 0 (with multiplicity 11), 4 and  $-3$ .

Eigenvalues for 0 are  $\vec{e}_1 - \vec{e}_i (i = 2, \dots, 12)$ ,

$$E_4 = \text{span} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 4 \end{bmatrix} \text{ (12 ones)}, E_{-3} = \text{span} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ -3 \end{bmatrix} \text{ (12 ones)}$$

so

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \end{bmatrix}$$

diagonalizes  $A$ , and  $D = S^{-1}AS$  will have all zeros as entries except  $d_{12, 12} = 4$  and  $d_{13, 13} = -3$ .

8.1.42 The matrix  $A$  is conjugated to  $\begin{bmatrix} 8 & 0 \\ 1 & 0 \end{bmatrix}$  by  $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} / \sqrt{5}$ . Now,  $B =$

$$S \begin{bmatrix} 8 & 0 \\ 1 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} / 5.$$