Solutions

7.6.**16** The eigenvalues are $4 \pm i$. We have $a = 4, b = 1$. The matrix is conjugated to $S^{-1}AS = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ with $S = \begin{bmatrix} 2-i & 2 \\ 1+i & 2 \end{bmatrix}$, which contains the eigenvectors of A as columns.

7.6.**12** $\lambda_{1,2} = 0.6 \pm ik$ so $\vec{0}$ is a stable equilibrium if $|\lambda_1| = |\lambda_2| = \sqrt{0.36 + k^2} < 1$ i.e. if $k^2 < 0.64$ or $|k| < 0.8$.

7.6.**16** $\lambda_{1,2} = \frac{2 \pm \sqrt{1+30k}}{10}$ so $|2 \pm \sqrt{1+30k}|$ must be less than 10. $\lambda_{1,2}$ are *real* if $k \geq -\frac{1}{30}$. In this case it is required that $2 + \sqrt{1+30k} < 10$ and $-10 < 2 - \sqrt{1+30k}$, which means that $\sqrt{1+30k} < 8$ or $k < \frac{21}{10}$.

$\lambda_{1,2}$ are *complex* if $k < -\frac{1}{30}$. Here it is required that $4 + (-1 - 30k) < 100$ or $k > -\frac{97}{30}$. Overall, $\vec{0}$ is a stable equilibrium if $-\frac{97}{30} < k < \frac{21}{10}$.

7.6.**24** $\lambda_{1,2} = -0.8 \pm 0.6i, r = 1, \theta = \pi - \arctan\left(\frac{.6}{.8}\right) \approx 2.5$ (second quadrant)

$[\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\vec{x}(t) = \begin{bmatrix} -5\sin(\theta t) \\ \cos(\theta t) - 3\sin(\theta t) \end{bmatrix}$, an ellipse, as shown in Figure 7.41.

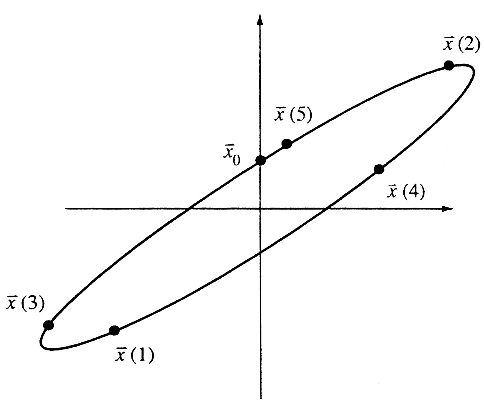

Figure 1: for Problem 7.6.24.

7.6.**34** a If $|\det A| = |\lambda_1 \lambda_2 \cdots \lambda_n| = |\lambda_1 \lambda_2| \cdots |\lambda_n| > 1$ then at least one eigenvalue is greater than one in modulus and the zero state fails to be stable.

b If $|\det A| = |\lambda_1||\lambda_2|\cdots|\lambda_n| < 1$ we cannot conclude anything about the stability of $\vec{0}$.

$|2||0.1| < 1$ and $|0.2||0.1| < 1$ but in the first case we would not have stability, in the second case we would.

7.6.**42** a $x(t+1) = x(t) - ky(t)$

$y(t+1) = kx(t) + y(t) = kx(t) + (1-k^2)y(t)$ so $\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

b $f_A(\lambda) = \lambda^2 - (2-k^2)\lambda + 1 = 0$

The discriminant is $(2-k^2)^2 - 4 = -4k^2 + k^4 = k^2(k^2-4)$, which is negative if k is a small positive number ($k < 2$). Therefore, the eigenvalues are complex. By Theorem 7.6.4 the trajectory will be an ellipse, since $\det(A) = 1$.

Solutions

7.TF :: 1 :: :: F; If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then eigenvalue 1 has geometric multiplicity 1 and algebraic multiplicity. 2.

7.TF :: 2 :: :: T, by Theorem 7.4.3

7.TF :: 3 :: :: T; $A = AI_n = A[\vec{e}_1 \dots \vec{e}_n] = [\lambda_1\vec{e}_1 \dots \lambda_n\vec{e}_n]$ is diagonal.

7.TF :: 4 :: :: T; If $A\vec{v} = \lambda\vec{v}$, then $A^3\vec{v} = \lambda^3\vec{v}$.

7.TF :: 5 :: :: T; Consider a diagonal 5×5 matrix with only two distinct diagonal entries.

7.TF :: 6 :: :: F, by Theorem 7.2.7.

7.TF :: 7 :: :: T, by Summary 7.1.5

7.TF :: 8 :: :: T, by Theorem 7.2.4

7.TF :: 9 :: :: T, by Theorem 7.2.2

7.TF :: 10 :: :: T, by Definition 7.2.3

7.TF :: 11 :: :: T, by Example 6 of Section 7.5

7.TF :: 12 :: :: T; The geometric multiplicity of eigenvalue 0 is $\dim(\ker A) = n - \text{rank}(A)$.

7.TF :: 13 :: :: T; If $S^{-1}AS = D$, then $S^TA^T(S^T)^{-1} = D$.

7.TF :: 14 :: :: F; Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, for example.

7.TF :: 15 :: :: F; Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

7.TF :: 16 :: :: F; Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \alpha = 2, B = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \beta = 5$, for example. Then $\alpha\beta = 10$ isn't an eigenvalue of $AB = \begin{bmatrix} 8 & 0 \\ 0 & 15 \end{bmatrix}$.

7.TF :: 17 :: :: T; If $A\vec{v} = 3\vec{v}$, then $A^2\vec{v} = 9\vec{v}$.

7.TF :: 18 :: :: T; Construct an eigenbasis by concatenating a basis of V with a basis of $V^\perp$.

7.TF :: 19 :: :: T, by Theorem 7.5.5

7.TF :: 20 :: :: F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, for example.

7.TF :: 21 :: :: T; If $S^{-1}AS = D$, then $S^{-1}A^{-1}S = D^{-1}$ is diagonal.

7.TF :: 22 :: :: F; the equation $\det(A) = \det(A^T)$ holds for all square matrices, by Theorem 6.2.1.

7.TF :: 23 :: :: T; The sole eigenvalue, 7, must have geometric multiplicity 3.

7.TF :: 24 :: :: F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, with $A+B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, for example.

7.TF :: 25 :: :: F; Consider the zero matrix.

7.TF :: 26 :: :: T; If $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$, then $(A+B)\vec{v} = A\vec{v}+B\vec{v} = \alpha\vec{v}+\beta\vec{v} = (\alpha+\beta)\vec{v}$.

7.TF :: 27 :: :: F; Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, with $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

7.TF :: 28 :: :: T, by Theorem 7.5.5

7.TF :: 29 :: :: F; Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, for example.

7.TF :: 30 :: :: F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, with $AB = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, for example.

7.TF :: 31 :: :: T, Consider the proof of Theorem 7.3.4a.

7.TF :: 32 :: :: T; An eigenbasis for A is an eigenbasis for $A + 4I_4$ as well.

7.TF :: 33 :: :: F; Consider the identity matrix.

7.TF :: 34 :: :: T; Both A and B are similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, by Theorem 7.4.1

7.TF :: 35 :: :: F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for example.

7.TF :: 36 :: :: F; Consider $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

7.TF :: 37 :: :: F; Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example.

7.TF :: 38 :: :: T; A nonzero vector on L and a nonzero vector on $L^\perp$ form an eigenbasis.

7.TF :: 39 :: :: T; The eigenvalues are 3 and -2.

7.TF :: 40 :: :: T, We will us Theorem 7.3.7 throughout: The geometric multiplicity of an eigenvalue is $\leq$ its algebraic multiplicity.

Now let's show the contrapositive of the given statement: If the geometric multiplicity of some eigenvalue is less than its algebraic multiplicity, then the matrix A fails to be diagonalizable. Indeed, in this case the sum of the geometric multiplicities of all the eigenvalues is less than the sum of their algebraic multiplicities, which in turn is $\leq n$ (where A is an $n \times n$ matrix). Thus the geometric multiplicities do not add up to n, so that A fails to be diagonalizable, by Theorem 7.3.4b.

7.TF :: 41 :: :: F; Consider a rotation through $\pi/2$.

7.TF :: 42 :: :: T; Suppose $\begin{bmatrix} A & A \\ 0 & A \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} A(\vec{v}+\vec{w}) \\ A\vec{w} \end{bmatrix} = \begin{bmatrix} \lambda\vec{v} \\ \lambda\vec{w} \end{bmatrix}$ for a nonzero vector $\begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix}$. If $\vec{w}$ is nonzero, then it is an eigenvector of A with eigenvalue λ; otherwise $\vec{v}$ is such an eigenvector.

7.TF :: 43 :: :: F; Consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

7.TF :: 44 :: :: T; Note that $S^{-1}AS = D$, so that $D^4 = S^{-1}A^4S = S^{-1}0S = 0$, and therefore $D = 0$ (since D is diagonal) and $A = SDS^{-1} = 0$.

7.TF :: 45 :: :: T; There is an eigenbasis $\vec{v}_1, \dots, \vec{v}_n$, and we can write $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. The vectors $c_i\vec{v}_i$ are either eigenvectors or zero.

7.TF :: 46 :: :: T; If $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$, then $AB\vec{v} = \alpha\beta\vec{v}$.

7.TF :: 47 :: :: T, by Theorem 7.3.6a

7.TF :: 48 :: :: F; Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, for example.

7.TF :: 49 :: :: T; Recall that the rank is the dimension of the image. If $\vec{v}$ is in the image of A, then $A\vec{v}$ is in the image of A as well, so that $A\vec{v}$ is parallel to $\vec{v}$.

7.TF :: 50 :: :: F; Consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

7.TF :: 51 :: :: T; If $A\vec{v} = \lambda\vec{v}$ for a nonzero $\vec{v}$, then $A^4\vec{v} = \lambda^4\vec{v} = \vec{0}$, so that $\lambda^4 = 0$ and $\lambda = 0$.

7.TF :: 52 :: :: F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, for example.

7.TF :: 53 :: :: T; If the eigenvalue associated with $\vec{v}$ is $\lambda = 0$, then $A\vec{v} = \vec{0}$, so that $\vec{v}$ is in the kernel of A; otherwise $\vec{v} = A\left(\frac{1}{\lambda}\vec{v}\right)$, so that $\vec{v}$ is in the image of A.

7.TF :: 54 :: :: T; either there are two distinct real eigenvalues, or the matrix is of the form kI_2.

7.TF :: 55 :: :: T; Either $A\vec{u} = 3\vec{u}$ or $A\vec{u} = 4\vec{u}$.

7.TF :: 56 :: :: T; Note that $(\vec{u}\vec{u}^T)\vec{u} = \|\vec{u}\|^2\vec{u}$.

7.TF :: 57 :: :: T; Suppose $A\vec{v}_i = \alpha_i\vec{v}_i$ and $B\vec{v}_i = \beta_i\vec{v}_i$, and let $S = [\vec{v}_1 \ldots \vec{v}_n]$.
Then $ABS = BAS = [\alpha_1\beta_1\vec{v}_1 \ldots \alpha_n\beta_n\vec{v}_n]$, so that $AB = BA$.

7.TF :: 58 :: :: T; Note that a nonzero vector $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if (and only if) $A\vec{v} = \begin{bmatrix} ap+bq \\ cp+dq \end{bmatrix}$ is parallel to $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$, that is, if $\det \begin{bmatrix} p & ap+bq \\ q & cp+dq \end{bmatrix} = 0$. Check that this is the case if (and only if) $\vec{v}$ is an eigenvector of adj(A) (use the same criterion).

Solutions