Solutions

7.4.**16** Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

If we let $S = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7.4.**30** First we observe that all the eigenspaces of $A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ are one-dimensional, regardless of the value of a, since $\mathrm{rref}(A - \lambda I_3)$ is of the form $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix}$ for all λ. Thus A is diagonalizable if and only if there are three distinct real eigenvalues. The characteristic polynomial of A is $-\lambda^3 + 3\lambda + a$. Thus the eigenvalues of A are the solutions of the equation $\lambda^3 - 3\lambda = a$. See Figure 7.24 with the function $f(\lambda) = \lambda^3 - 3\lambda$; using calculus, we find the local maximum $f(-1) = 2$ and the local minimum $f(1) = -2$. To count the distinct eigenvalues of A, we have to examine how many times the horizontal line $y = a$ intersects the graph of $f(\lambda)$. The answer is three if $|a| < 2$, two if $a = \pm 2$, and one if $|a| > 2$. Thus A is diagonalizable if and only if $|a| < 2$, that is, $-2 < a < 2$.

7.4.**36** Yes. The matrices $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ both have the eigenvalues 3 and 2, so that each of them is similar to the diagonal matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, by Algorithm 7.4.4. Thus $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, by parts b and c of Theorem 3.4.6.

7.4.**56** The hint shows that matrix $M = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ is similar to $N = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$; thus matrices M and N have the same characteristic polynomial, by Theorem 7.3.6a. Now $f_M(\lambda) = \det \begin{bmatrix} AB - \lambda I_n & 0 \\ B & -\lambda I_n \end{bmatrix} = (-\lambda)^n \det(AB - \lambda I_n) = (-\lambda)^n f_{AB}(\lambda)$. To understnd the second equality, consider Theorem 6.1.5. Likewise, $f_N(\lambda) = (-\lambda)^n f_{BA}(\lambda)$. It follows that $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^n f_{BA}(\lambda)$ and therefore $f_{AB}(\lambda) = f_{BA}(\lambda)$, as claimed.

7.4.**60** The matrix A satisfies $A^2 = 0$. Use the method outlined in Exercise 7.4.58. We start with a vector $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the image of A. Since $\vec{v}_1 =$ first column of $A = A\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ we can let $\vec{v}_2 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Finally, let $\vec{v}_3 = \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix}$ and other vector in the kernel (use Kyle numbers!). Thus $S = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$ will do the job but there are other answers.