Solutions

7.3.10 $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$, $E_1 = \text{span}\begin{bmatrix}1\\0\\0\end{bmatrix}, E_0 = \text{span}\begin{bmatrix}0\\0\\1\end{bmatrix}$

No eigenbasis

7.3.18 $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 1$, $E_0 = \text{span}(\vec{e}_1, \vec{e}_3)$, $E_1 = \text{span}(\vec{e}_2)$

No eigenbasis

7.3.26 Note that $f_A(0) = \det(A - 0I_6) = \det(A)$ is negative. Since $\lim_{\lambda\to\infty} f_A(\lambda) = \infty$, there must be a positive root, by the Intermediate Value Theorem (see Exercise 2.2.47c). Therefore, the matrix A has a positive eigenvalue. See Figure 7.18.

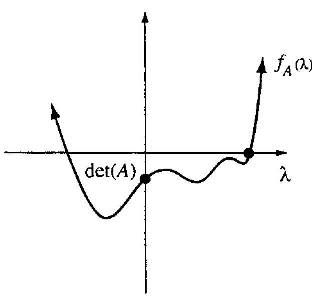

Figure 1: for Problem 7.3.26.

7.3.28 Since $J_n(k)$ is triangular, its eigenvalues are its diagonal entries, hence its only eigenvalue is k. Moreover,

$$E_k = \ker(J_n(k) - kI_n) = \ker\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\0 & 0 & 1 & & \vdots\\ \vdots & \vdots & 0 & & \vdots\\ \vdots & \vdots & \vdots & & 1\\ 0 & 0 & 0 & & 0\end{bmatrix} = \text{span}\ (\vec{e}_1).$$

The geometric multiplicity of k is 1 while its algebraic multiplicity is n.

7.3.44 a $a_{11} = 0.7$ means that only 70% of the pollutant present in Lake Sils at a given time is still there a week later; some is carried down to Lake Silvaplana by the river Inn, and some is absorbed or evaporates. The other diagonal entries can be interpreted analogously: $a_{21} = 0.1$ means that 10% of the pollutant present in Lake Sils at any given time can be found in Lake Silvaplana a week later, carried down by the river Inn. The significance of the coefficient $a_{32} = 0.2$ is analogous; $a_{31} = 0$ means that no pollutant is carried down from Lake Sils to Lake St. Moritz in just one week. The matrix is lower triangular since no pollutant is carried from Lake Silvaplana to Lake Sils. The river Inn would have to flow the other way.

b The eigenvalues of A are 0.8, 0.6, 0.7, with corresponding eigenvectors

$$\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}.$$

$$\vec{x}(0) = \begin{bmatrix}100\\0\\0\end{bmatrix} = 100\begin{bmatrix}0\\0\\1\end{bmatrix} - 100\begin{bmatrix}0\\1\\-1\end{bmatrix} + 100\begin{bmatrix}1\\1\\-2\end{bmatrix},$$

so $\vec{x}(t) = 100(0.8)^t\begin{bmatrix}0\\0\\1\end{bmatrix} - 100(0.6)^t\begin{bmatrix}0\\1\\-1\end{bmatrix} + 100(0.7)^t\begin{bmatrix}1\\1\\-2\end{bmatrix}$ or

$x_1(t) = 100(0.7)^t$

$x_2(t) = 100(0.7)^t - 100(0.6)^t$

$x_3(t) = 100(0.8)^t + 100(0.6)^t - 200(0.7)^t$. See Figure 7.22.

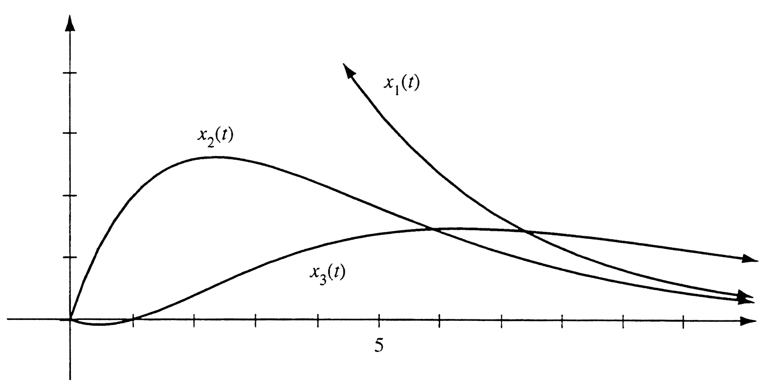

Figure 2: for Problem 7.3.44b.

Using calculus, we find that the function $x_2(t) = 100(0.7)^t - 100(0.6)^t$ reaches its maximum at $t \approx 2.33$. Keep in mind, however, that our model holds for integer t only.