## Solutions

7.1.**14** We want to find all $4 \times 4$ matrices $A$ such that $A\vec{e}_2 = \lambda\vec{e}_2$, i.e. the second column of $A$ must be of the form $\begin{bmatrix} 0 \\ \lambda \\ 0 \\ 0 \end{bmatrix}$, so $A = \begin{bmatrix} a & 0 & c & d \\ e & \lambda & f & g \\ h & 0 & i & j \\ k & 0 & l & m \end{bmatrix}$.

7.1.**36** We want $A$ such that $A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, i.e. $A\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}$, so

$$A = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}.$$

7.2.**12** $f_A(\lambda) = \lambda(\lambda+1)(\lambda-1)^2$ so $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 1$ (Algebraic multiplicity 2).

7.2.**28** a $w(t+1) = 0.8w(t) + 0.1m(t)$

$m(t+1) = 0.2w(t) + 0.9m(t)$

so $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ which is a regular transition matrix since its columns sum to 1 and its entries are positive.

b The eigenvectors of $A$ are $\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_1 = 1$, and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $\lambda_2 = 0.7$.

$\vec{x}_0 = \begin{bmatrix} 1200 \\ 0 \end{bmatrix} = 400\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $\vec{x}(t) = 400\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800(0.7)^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or

$w(t) = 400 + 800(0.7)^t$

$m(t) = 800 - 800(0.7)^t$.

c As $t \to \infty$, $w(t) \to 400$ so Wipfs won't have to close the store.

7.2.**40** Let the entries of $A$ be $a_{ij}$ and the entries of $B$ be $b_{ij}$.

Now, $\text{tr}(AB) = (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + \cdots + a_{2n}b_{n2}) + \cdots + (a_{n1}b_{1n} + \cdots + a_{nn}b_{nn})$. This is the sum of all products of the form $a_{ij}b_{ji}$.

We see that $\text{tr}(BA) = (b_{11}a_{11} + \cdots + b_{1n}a_{n1}) + \cdots + (b_{n1}a_{1n} + \cdots + b_{nn}a_{nn})$, which also is the sum of all products of the form $b_{ji}a_{ij} = a_{ij}b_{ji}$. Thus, $\text{tr}(AB) = \text{tr}(BA)$.