Solutions

6.2.**6** $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix} \begin{matrix} \\ -I \\ -I \\ -I \end{matrix} \rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \text{swap:} \\ II \leftrightarrow III \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \div -2 \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \\ \\ \\ +2II \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix} \begin{matrix} \\ \\ \\ -2III \end{matrix} \rightarrow \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$\det(A) = -\frac{1}{2}(-1)\det(B) = -72$, by Algorithm 6.2.5b.

6.2.**8** After four row swaps, we end up with an upper triangular matrix B with all 1's along the diagonal, except for a 2 in the bottom right corner. Now $\det(A) = (-1)^4 \det(B) = 2$, by Algorithm 6.2.5b.

6.2.**50** There are many ways to do this problem; here is one possible approach:

Subtracting the second to last row from the last, we can make the last row into

$[0 \quad 0 \quad \cdots \quad 0 \quad 1]$.

Now expanding along the last row we see that $\det(M_n) = \det(M_{n-1})$.

Since $\det(M_1) = 1$ we can conclude that $\det(M_n) = 1$ for all n.

6.2.**66** a Using Laplace expansion with respect to the first row we get $d_n = d_{n-1} - d_{n-2}$. Proof

$$d_n = \det(M_n) = \det(M_{n-1}) - \det \begin{bmatrix} 1 & * \\ 0 & M_{n-2} \end{bmatrix} = \det(M_{n-1}) - \det(M_{n-2}) = d_{n-1} - d_{n-2}.$$

b $d_1 = 1, d_2 = 0, d_3 = -1, d_4 = -1, d_5 = 0, d_6 = 1, d_7 = 1, d_8 = 0.$

c $d_4 = -d_1$ and $d_5 = -d_2$, the formula $d_{n+3} = -d_n$ holds for all positive integers n. (One can give a formal prove by induction). Now $d_{n+6} = -d_{n+3} = -(-d_n) = d_n$ for all positive integers n, meaning that the sequence d_n has a period of six.

d $d_{100} = d_4 = -d_1 = -1.$

6.3.**12** Denote the columns by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$. From Theorem 6.3.4 and Exercise 6.3.8 we know that $|\det(A)| \leq \|\vec{v}_1\|\|\vec{v}_2\|\ \|\vec{v}_3\|\|\vec{v}_4\|$; equality holds if the columns are orthogonal. Since the entries of the $\vec{v}_i$ are 0, 1, and -1, we have $\|\vec{v}_i\| \leq \sqrt{1+1+1+1} = 2$. Therefore, $|\det A| \leq 16$.

To build an example where $\det(A) = 16$ we want all 1's and -1's as entries, and the columns need to be orthogonal. A little experimentation produces $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ (there are other solutions). Note that we need to *check* that $\det(A) = 16$ (and not -16).