

Solutions

6.1.38 By Theorem 6.1.5, the determinant is equal to $\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 2 & 1 & 2 \end{bmatrix} \det \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$$= (9 + 16 - 12 - 4)(36 - 25) = 99.$$

6.1.42 $\det \begin{pmatrix} 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 9 & 7 & 9 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 3 & 4 & 5 & 8 & 5 \end{pmatrix} = 540$. There is only one pattern which has a nonzero product, $(a_{13}, a_{24}, a_{32}, a_{45}, a_{51}) = (2, 2, 9, 5, 3)$. There are six inversions and its signature is therefore 1. The determinant is $3 * 9 * 2 * 2 * 5 = 540$.

6.1.48 Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $\det(A) = \det(B) = \det(C) = \det(D) = 0$ hence $\det(A) \det(D) - \det(B) \det(C) = 0$ but $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = -1$.

6.1.54 The determinant of the matrix

$$A = \begin{bmatrix} 1 & 1000 & 2 & 3 & 4 \\ 5 & 6 & 7 & 1000 & 8 \\ 1000 & 9 & 8 & 7 & 6 \\ 1 & 2 & 1000 & 3 & 4 \end{bmatrix}$$

is positive, because there is a dominant pattern containing all the 1000's with 4 inversions. This pattern contributes $(1000)^5 = 10^{15}$ to the determinant. There are $5! - 1 = 119$ other patterns with at most 3 entries being 1000, the others being ≤ 9 . Thus the product associated with each of those pattern is less than $(1000)^3(10)^2 = 11^{11}$. Now $\det(A) > 10^{15} - 119 * 10^{11} > 0$.

6.1.56 a $\det(M_4) = \det(M_5) = 1$. $\det(M_2) = \det(M_3) = \det(M_6) = \det(M_7) = -1$.

b $d_1 = 1, d_2 = -1, d_3 = -1, d_4 = 1, d_5 = 1, d_6 = -1, d_7 = -1, d_8 = 1$. We notice an oscillation between -1 and 1 with every other increase and that $d_{n+4} = d_n$. Lets prove this: there is only one pattern with a nonzero product, containing all the 1's. The number of inversions is $(n-1) + (n-2) + \dots + 2 + 1 = \sum_{k=1}^{n-1} k = n(n-1)/2$. This number is

even if either n or $n - 1$ is divisible by 4, that is, for $n = 4, 5, 8, 9, 12, 13, \dots$. We can write $\det(M_n) = (-1)^{n(n-1)/2}$.

c By the periodicity in part *b*, we see that d_{100} will be equal to $d_{100-24(4)} = d_4 = 1$.