

Solutions

3.4.12 Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

3.4.18 Here, \vec{x} is not in V , as we find an inconsistency while attempting to solve the system.

3.4.26 Let's build B "column-by-column":

$$\begin{aligned} B &= [[T(\vec{v}_1)]_{\mathcal{B}} | [T(\vec{v}_2)]_{\mathcal{B}}] \\ &= \left[\left[\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[\begin{bmatrix} 2 \\ 8 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 6 & 4 \\ -4 & -3 \end{bmatrix}. \end{aligned}$$

3.4.32 Here we will build B column-by-column:

$$\begin{aligned} B &= [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}] \\ &= [[\vec{v}_1 \times \vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_2 \times \vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_3 \times \vec{v}_3]_{\mathcal{B}}] = [[-\vec{v}_2]_{\mathcal{B}} \quad [\vec{v}_1]_{\mathcal{B}} \quad \vec{0}], \text{ since all three are perpen-} \\ &\text{dicular unit vectors.} \end{aligned}$$

$$\text{So, } B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.4.42 From Exercise 38, we deduce that one of our vectors should be perpendicular to this plane, while two should fall inside it. Finding the perpendicular is not difficult: we simply

take the coefficient vector: $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. Then we add two linearly independent vectors on the

plane, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, for instance. These three vectors form one possible basis.

3.4.56 Let $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$ be the desired basis. We know

$$\begin{aligned} v_1 + 2v_2 &= \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ 3v_1 + 4v_2 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

With $S = [\vec{v}_1 \vec{v}_2]$ this can be written as $S \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$. [Alternatively, by Theorem 3.4.1 we can get this directly from $S \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $S \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.]

Solving for S gives $S = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 7/2 \\ -7 & 6 \end{bmatrix}$. The desired basis is $\begin{bmatrix} -4 \\ -7 \end{bmatrix}, \begin{bmatrix} 7/2 \\ 6 \end{bmatrix}$.

3.4.64 If b and c are both zero, then the given matrices are equal, so that they are similar, by Theorem 3.4.6.a. Let's now assume that at least one of the scalars b and c is nonzero; reversing the roles of b and c if necessary, we can assume that $c \neq 0$.

Let's find the matrices $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, or

$\begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix} = \begin{bmatrix} ax + by & cx + dy \\ az + bt & cz + dt \end{bmatrix}$. The solutions are of the form

$S = \begin{bmatrix} \frac{(a-d)z+b}{c} & z \\ z & t \end{bmatrix}$, where z and t are arbitrary constants. Since there are *invertible*

solutions S (for example, let $z = 1, t = 0$), the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ are indeed similar.