Solutions

3.3.**18** This matrix is already in rref, and we see that there are two columns without leading ones. These will be our redundant columns. Thus we see

$$\begin{matrix} 2 & 1 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} -1 & 0 & 5 & -1 & 0 \end{matrix}$$
$$\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right)$ is a basis of the kernel, and $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ is a basis of the image.

3.3.**26** a We notice that each of the six matrices has two identical columns. In matrices C and L, the second column is identical to the third, so that $\ker(C) = \ker(L) = \text{span} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. In matrices H, T, X and Y, the first column is identical to the third, so that $\ker(H) = \ker(T) = \ker(X) = \ker(Y) = \text{span} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Thus, only L has the same kernel as C.

b We observe that each of the six matrices in the list has two identical rows. For example, the first and the last row of matrix C are identical, so that any vector $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in $\text{im}(C)$ will satisfy the equation $y_1 = y_3$. We can conclude that $\text{im}(C) = \text{im}(H) = \text{im}(X) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_3 \right\}$, $\text{im}(L) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_2 \right\}$, and $\text{im}(T) = \text{im}(Y) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_2 = y_3 \right\}$.

c Our discussion in part *b* shows that the answer is matrix L.

3.3.**32** We need to find all vectors $\vec{x}$ in $\mathbb{R}^4$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$.

This amounts to solving the system $\begin{bmatrix} x_1 & & - & x_3 & + & x_4 & = & 0 \\ & x_2 & + & 2x_3 & + & 3x_4 & = & 0 \end{bmatrix}$, which in turn amounts to finding the kernel of $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

Using Kyle Numbers, we find the basis $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

3.3.**36** No; if $\text{im}(A) = \ker(A)$ for an $n \times n$ matrix A, then $n = \dim(\ker(A)) + \dim(\text{im}(A)) = 2\dim(\text{im}(A))$, so that n is an even number.

3.3.**38** a The rank of a 3×5 matrix A is 0,1,2, or 3, so that $\dim(\ker(A)) = 5 - \text{rank}(A)$ is 2,3,4, or 5.

b The rank of a 7×4 matrix A is at most 4, so that $\dim(\text{im}(A)) = \text{rank}(A)$ is 0,1,2,3, or 4.

3.3.**72** $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

By Exercises 3.3.70 and 3.3.71a, $[1\ 0\ -1\ -2]$, $[0\ 1\ 2\ 3]$ is a basis of the row space of A.

3.3.**84** Same answer as Exercise 3.3.85.