Name:

MWF10 Oliver Knill
MWF11 Anand Patel

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If x^* is the least squares solution of $Ax = b$, then $||b||^2 = ||Ax^*||^2 + ||b - Ax^*||^2$.

Solution:
This is Pythagoras applied to the orthogonal vectors $b - Ax^*, Ax^*$.

2) T F Similar matrices have the same determinant.

Solution:
$\det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \det(A)$.

3) T F If λ is an eigenvalue of A, then λ^3 is an eigenvalue of A^3.

Solution:
$A\vec{v} = c\vec{v}$, then $A^3\vec{v} = c^3\vec{v}$.

4) T F A shear in the plane is not diagonalizable.

Solution:
Indeed, the shear in the plane is the prototype of a nondiagonalizable transformation.

5) T F If A is invertible, then A and A^{-1} have the same eigenvectors.

Solution:
Yes, if $A\vec{v} = \lambda\vec{v}$, then $A^{-1}\vec{v} = \lambda^{-1}\vec{v}$.

6) T F If A is a 3×3 matrix for which every entry is 1, then $\det(A) = 1$.

Solution:
The kernel is nontrivial, contains for example $[1, -1, 0]^T$.

7) T F If $\vec{v}$ is an eigenvector of A and of B and A is invertible, then $\vec{v}$ is an eigenvector of $3A^{-1} + 2B$.

Solution:
$\vec{v}$ is also an eigenvector of A^{-1}.

8) T F $\det(-A) = \det(A)$ for every 5×5 matrix A.

Solution:
For odd n, an $n \times n$ matrix satisfies $\det(-A) = -\det(A)$.

9) T F If $\vec{v}$ is an eigenvector of A and an eigenvector of B and A is invertible, then $\vec{v}$ is an eigenvector of $A^{-3}B^2$.

Solution:
Again, one can add and multiply matrices with the same eigenvector and get matrices with the same eigenvector.

10) T F If a 11×11 matrix has the eigenvalues $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, then A is diagonalizable.

Solution:
All eigenvalues are different.

11) T F For any $n \times n$ matrix, the matrix A has the same eigenvectors as A^T.

Solution:
The same eigenvalues yes, but not the same eigenvectors.

12) T F If $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector of length 1, then $\vec{v}\vec{v}^T$ is a diagonalizable 3×3 matrix.

Solution:
The matrix is a projection matrix onto a one dimensional line. In suitable coordinates, this matrix has only 1 or 0 in the diagonal.

13) [T] [F] The span of m orthonormal vectors is m-dimensional.

Solution:
Yes, orthogonal vectors can not be linearly dependent.

14) [T] [F] A square matrix A can always be expressed as the sum of a symmetric matrix and a skew-symmetric matrix as follows $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

Solution:
Indeed, $A + A^T$ is symmetric and $A - A^T$ is skew symmetric and the sum as given holds.

15) [T] [F] There exists an invertible $n \times n$ matrix A which satisfies $A^T A = A^2$, but A is not symmetric.

Solution:
Because A is invertible, one can multiply the equation from the right with A^{-1} and gets $A^T = A$.

16) [T] [F] If two $n \times n$ matrices A and B commute, then $(A^T)^2$ commutes with $(B^3)^T$.

Solution:
If A and B commute, then A^T commutes with B^T and therefore $(A^n)^T$ commutes with $(B^m)^T$ for all m, n. Note that A^T does not necessarily commute with B.

17) [T] [F] $-AA^T$ is skew-symmetric for every $n \times n$ matrix A.

Solution:
$B = -AA^T$ satisfies $B^T = -AA^T = B$ so $-AA^T$ is symmetric.

18) T F — A matrix which is obtained from the identity matrix by an arbitrary number of switching of rows or columns is an orthogonal matrix.

Solution:
Indeed, switching of columns or rows does not change the orthogonality of each pair of vectors.

19) T F — There exists a real 3×3 matrix A which satisfies $A^4 = -I_3$.

Solution:
Take determinants to see that this is not possible.

20) T F — Given 5 data points $(x_1, y_1), ..., (x_5, y_5)$, then a best fit with a polynomial $a + bt + ct^2 + dt^3 + et^4 + ft^5$ is possible in a unique way.

Solution:
There are 6 variables and 5 conditions to be satisfied, one has in this case infinitely many possibilities to fit the data without error.

Problem 2) (10 points)

Check the boxes of all matrices which have zero determinants. You don't have to give justifications.

a) $\square$ $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

b) $\square$ $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$

c) $\square$ $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

d) $\square$ $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}$

e) $\square$ $A = \begin{bmatrix} 10^{100} & 1 & 1 & 1 \\ 1 & 10^{100} & 1 & 1 \\ 1 & 1 & 10^{100} & 0 \\ 1 & 0 & 1 & 10^{100} \end{bmatrix}$

f) $\square$ $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

g) $\square$ $A = \begin{bmatrix} 11 & 10 & 8 & 5 \\ 9 & 7 & 4 & 0 \\ 6 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

h) $\square$ $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

i) $\square$ $A = \begin{bmatrix} 1 & 1 & 7 & 0 \\ 1 & 6 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

j) $\square$ $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$

Solution:
b),c),f),j) have zero determinant. Some hints:
a) One pattern is nonzero only.
b) The differences between consecutive rows is the same vector.
c) Two rows are identical.
d) One pattern is nonzero only.
e) One pattern dominates clearly.
f) Partitioned matrix, the matrix at the lower right corner is noninvertible.
g) One pattern, the anti-diagonal is nonzero only.
h) You have constructed this matrix as the one which maximizes the determinant among all matrices with entries $0, -1, 1$.
i) One pattern only. Can also be seen as a partitioned matrix.
j) Parallel columns.

Problem 3) (10 points)

a) Find all (possibly complex) eigenvalues and eigenvectors of the matrix $Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b) Verify that Q^T has the same eigenvectors as Q.

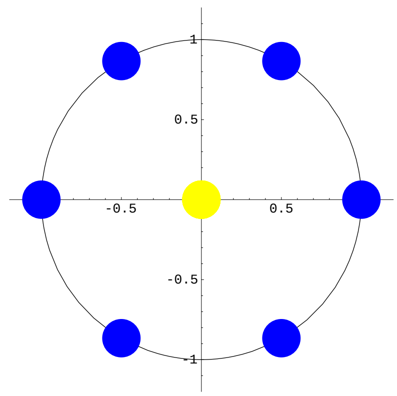
c) Find a diagonal matrix B which is similar to the symmetric matrix

$$A = Q + Q^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

State algebraic and geometric multiplicities of the eigenvalues.

Solution:

a) The characteristic polynomial is $\lambda^6 - 1$. The eigenvalues are the $6'th$ roots of 1 and are of the form $\lambda_k = e^{i2\pi k/6}$. For each of these eigenvalues λ, there is an eigenvector $\vec{v}_\lambda = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \\ \lambda^5 \end{bmatrix}$.



($Q\vec{v} = \begin{bmatrix} a_5 \\ 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \lambda\vec{v}$ implies $a_1 = \lambda, a_2 = \lambda a_1, a_3 = \lambda a_2, a_4 = \lambda a_3, a_5 = \lambda a_4$.

b) Because $Q^T = Q^{-1}$, both Q and Q^T have the same eigenvector: if $Q\vec{v} = \lambda\vec{v}$, then $Q^T\vec{v} = Q^{-1}\vec{v} = \lambda^{-1}\vec{v}$.
So, if λ is an eigenvalue with eigenvector $\vec{v}$, then $\lambda + \lambda^{-1}$ is an eigenvalue of A with eigenvector $\vec{v}$.
c) The matrix A has the eigenvalue $2\cos(2\pi k/6)$. These are $-2, 2$ with algebraic multiplicity 1 and $-1, 1$ with algebraic multiplicity 2.

The matrix is similar to $B = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Remark (but this was not asked).
While the complex eigenvectors $\vec{v}_\lambda$ are also eigenvectors of A, there are also **real** eigenvectors of A. $\lambda_1 = -2, \vec{v} = [-1, 1, -1, 1, -1, 1]$, $\lambda_2 = 2, \vec{v} = [1, 1, 1, 1, 1, 1]$, $\lambda_3 = -1, \vec{v} = [-1, 0, 1, -1, 0, 1]$ and $\vec{v} = [0, 1, -1, 0, 1, -1]$ $\lambda_4 = 1, \vec{v} = [1, 0, -1, -1, 0, 1]$ and $\vec{v} = [0, -1, -1, 0, 1, 1]$.

Problem 4) (10 points)

a) Let A be a $n \times n$ matrix such that $A^2 = 2A - I$. What are the possible eigenvalues of A?

b) Let A be a real $n \times n$ matrix such that $A^4 = -I_n$. Show that n must be even.

Solution:
a) λ has to be a root of the polynomial $\lambda^2 = 2\lambda - 1$. This means 1 is the only possible eigenvalue.
b) Look at the determinant of A. If n is odd, the determinant of the left hand side is nonnegative, on the right hand side it is negative.

Problem 5) (10 points)

Find the function $y = f(x) = a\cos(\pi x) + b\sin(\pi x)$, which best fits the data

x	y
0	1
1/2	3
1	7

Solution:
We have to find the least square solution to the system of equations

$$\begin{aligned} 1a + 0b &= 1 \\ 0a + 1b &= 3 \\ -1a + 0b &= 7 \end{aligned}$$

which is in matrix form written as $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}.$$

Now $A^T\vec{b} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $(A^TA)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$ and $(A^TA)^{-1}A^T\vec{b}$ is $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$. The best fit is the function $\boxed{f(x) = -3\cos(\pi x) + 3\sin(\pi x)}$.

Problem 6) (10 points)

Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & 0 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & 0 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & 0 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & 0 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & 0 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & 0 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & 0 \end{bmatrix} .$$

Show your work carefully.

Solution:
First solution.
Adding the first row to the others gives a upper triangular matrix with entries $1, 2, 3, 4, 5, 6, 7, 8$ in the diagonal. The answer is $8! = 40'320$.
Second solution.
Look at the matrix as a partitioned matrix, which is made of 2×2 matrices. Notice, that most of these 2×2 matrices have zero determinant. Each of these can be replaced by the zero matrix without changing the determinant. We end up with a matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 \end{bmatrix}$$

which has determinant 8! too.
Third solution.
Look at the matrix as a partitioned matrix made up of 4 matrices of size 4×4. The anti diagonal matrices have zero determinant and can be replaced by the zero matrix without changing the determinant.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 4 & 0 & 0 & 0 & 0 \\ -1 & -2 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & -5 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & -5 & -6 & 0 & 8 \\ 0 & 0 & 0 & 0 & -5 & -6 & -7 & 0 \end{bmatrix}$$

One can apply now the same trick to the two 4×4 matrices, split it up into 2.

Problem 7) (10 points)

A discrete dynamical system is given by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x - 2y \\ -x + 3y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} .$$

Find a closed formula for $T^{100}(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$.

Solution:
The eigenvector to the eigenvalue 4 is $\vec{v} = [-1, 1]^T$. The eigenvector to the eigenvalue 1 is $\vec{w} = [2, 1]^T$. Because $[1, 2]^T = \vec{v} + \vec{w}$, we have $T^{100}(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = 4^{100} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1^{100} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Problem 8) (10 points)

a) Show that for an arbitrary matrix A for which $A^T A$ is invertible, the least squares solution of $A\vec{x} = \vec{b}$ simplifies to

$$\vec{x} = R^{-1} Q^T \vec{b} ,$$

if $A = QR$ is the QR decomposition of A.

b) Find the least square solution in the case $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ using this formula.

Solution:
a) This is a direct algebraic computation, using $Q^T Q = I$:

$$(A^T A)^{-1} A^T = (R^T Q^T Q R)^{-1} R^T Q^T = R^{-1} (R^T)^{-1} R^T Q^T = R^{-1} Q^T .$$

The QR decomposition is $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2^{-1/2} \\ 0 & 2^{-1/2} \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$.

b) Use a) to get $R^{-1} Q^T b = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$.

Problem 9) (10 points)

Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A with their algebraic multiplicities.

b) Find the geometric multiplicities of each of the eigenvalues.

c) Find all the eigenvectors.

d) What is $\det(A)$?

Solution:
Look at it as a partitioned matrix:
a) $2+\sqrt{3}, 2-\sqrt{3}, 3, 3, 3+i, 3-i$ are the eigenvalues.
b) The geometric multiplicities are 1 for all eigenvalues. Note that 3 appears twice and has the algebraic multiplicity 2 but the geometric multiplicity is 1.
c) The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ -\sqrt{3} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ i \\ 1 \end{bmatrix} v_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$$

The eigenvector v_3 is the only eigenvector to the eigenvalue 3.
d) The determinant is the product of the eigenvalues which is $\boxed{90}$.

Problem 10) (10 points)

a) (7 points) Find the QR decomposition of

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

then form the new matrix $T(A) = RQ$.

b) (3 points) Verify that for any invertible $n \times n$ matrix $A = QR$, the matrix $T(A) = RQ$ has the same eigenvalues as A.

c) Find the least square solution for the system $A\vec{x} = \vec{b}$ given by the equations

$$x + y = 4$$

$$\begin{aligned} y &= 2 \\ x &= -1\,. \end{aligned}$$

Solution:

a) $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. $R = \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$. $RQ = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$.

b) Take $S = Q$, then $ST(A)S^{-1} = QRQQ^{-1} = QR$, so that $T(A) = RQ$ is similar to $A = QR$ and has the same spectrum as A.

c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$. Form $\vec{x} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.