Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, **give details**. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

1)	T	F	Let A and B be two 2×2 matrices. Then A and B are similar if and only if they have the same trace and determinant.
2)	T	F	An orthogonal $n \times n$ matrix satisfies $AA^T = I_n$.
3)	T	F	The eigenvectors of a matrix A do not change under row reduction.
4)	T	F	If A is the matrix of a reflection at a line in space, then $\det(A^2 - I_3) = 0$.
5)	T	F	Every rotation matrix can be diagonalized over the reals.
6)	T	F	The recursion $x_{n+1} = x_n + x_{n-1} + x_{n-2} + x_{n-3}$ can be written as $\vec{v}(n+1) = A\vec{v}_n$ for a 4×4 matrix A and a vector $\vec{v}$.
7)	T	F	The are two rotations such that their product is an orthogonal projection.
8)	T	F	The nulletly of A is the same as the nulletly of A^T.
9)	T	F	For any matrix, we have $\det(A^5/2) = \det(A)^5/2$.
10)	T	F	The trace of a matrix is the sum of the eigenvalues
11)	T	F	There is a reflection for which the determinant is equal to 2.
12)	T	F	The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix}$.
13)	T	F	If two 2×2 matrices A and B have the same trace, determinant and geometric multiplicities of all eigenvalues are 1 then they are similar.
14)	T	F	If A has no kernel, then the least square solution of $Ax = b$ is unique.
15)	T	F	If $A^n\vec{v}$ and $A^{-n}\vec{v}$ both stay bounded for every vector $\vec{v}$, then all eigenvalues of A are located on the unit circle $\lvert\lambda\rvert = 1$ in the complex plane.
16)	T	F	If an orthogonal matrix Q is symmetric, then Q is diagonal.
17)	T	F	If $A = QR$ is the QR decomposition of a square matrix, then the eigenvalues of A are the diagonal entries of R.
18)	T	F	For every 2×2 matrix A, we have $A - I_n$ is similar to $A + I_n$.
19)	T	F	If A is similar to B then A^{-1} is similar to B^{-1}.
20)	T	F	For any 2×2 regular transition matrix, the trace determines the determinant.

Total

Problem 2) (10 points) No justifications are needed.

a) (6 points) Which of the following assertions are true?

true	false	
		A is similar to B
		A is similar to C
		A is similar to D

true	false	
		B is similar to C
		B is similar to D
		C is similar to D

$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

b) (4 points) Match the following matrices with sets of eigenvalues. You are told that there is a unique match. It is not always necessary to compute all the eigenvalues to do so.

Eneter i),ii),iii) or iv)	The matrix
	$A = \begin{bmatrix} -1 & -2 & 8 \\ -7 & -3 & 19 \\ -3 & -2 & 10 \end{bmatrix}$
	$A = \begin{bmatrix} 5 & -9 & -7 \\ 0 & 5 & 2 \\ 0 & 0 & 6 \end{bmatrix}$
	$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
	$A = \begin{bmatrix} 13 & 11 & 13 \\ -2 & -1 & -2 \\ -8 & -7 & -8 \end{bmatrix}$

i) $\{3, 2, 1\}$.

ii) $\{1, 0, 3\}$.

iii) $\{6, 5, 5\}$.

iv) $\{1, i, -i\}$.

Problem 3) (10 points) No justifications are needed

a) (5 points) Given a unit column vector written as a 4×1 matrix $A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$. Fill in the values of the determinants and eigenvalues of the following matrices and state whether they are symmetric

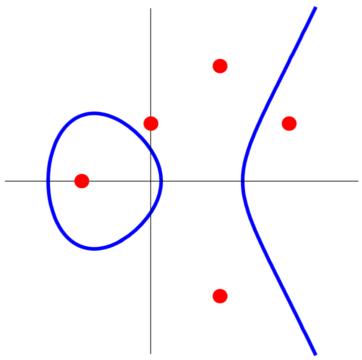
Matrix	Determinant	Eigenvalues	Symmetric? Yes/No
A^TA			
AA^T			
$A(A^TA)^{-1}A^T$			

b) (5 points) No explanations are necessary for this problem.

Which matrices are orthogonal, which matrices are anti-symmetric $A^T = -A$. Which matrices are projections? Check everything which applies. It is not excluded that you have to check several properties for each matrix.

	orthogonal	antisymmetric	projection	
1)				$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$
2)				$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3)				$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
4)				$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Problem 4) (10 points)



A curve of the form

$$y^2 = x^3 + ax + b$$

is called an **elliptic curve** in Weierstrass form. Elliptic curves are important in cryptography. Use data fitting to find the best parameters (a, b) for an elliptic curve given the following points:

$$\begin{array}{rcl} (x_1, y_1) & = & (1, 2) \\ (x_2, y_2) & = & (-1, 0) \\ (x_3, y_3) & = & (2, 1) \\ (x_4, y_4) & = & (0, 1) \end{array}$$

Hint. As usual, write down a system of equations $Ax = v$ with $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$. which you solve using the least square solution method.

Problem 5) (10 points)

The sequence $2, 1, -1, -5, -13, -29, -61, \ldots$ satisfies the recursion

$$x(n+1) = 3x(n) - 2x(n-1)$$

with $x(0) = 2, x(1) = 1$. Find $x(n)$.

Hint. As usual, we write the recursion first in the form $\vec{v}(n+1) = A\vec{v}(n)$, where A is a 2×2 matrix, where $\vec{v}(n) = \begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix}$ and where the initial condition is $\vec{v} = \vec{v}(0) = \begin{bmatrix} x(1) \\ x(0) \end{bmatrix}$.

Problem 6) (10 points)

a) (4 points) A 2×2 matrix A satisfies $\text{tr}(A^2) = 5$ and $\text{tr}(A) = 3$. Find $\det(A)$.

b) (3 points) A 2×2 matrix has two parallel columns and $\text{tr}(A) = 5$. Find $\text{tr}(A^2)$.

c) (3 points) A 2×2 matrix A has $\det(A) = 5$ and positive integer eigenvalues. What is the trace of A?

Problem 7) (10 points)

a) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1001 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1002 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 1003 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 1004 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 1005 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 & 1006 & 6 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 & 1007 \end{bmatrix}$$

b) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 & 1 & 4 \end{bmatrix}.$$

c) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 8 & 9 & 2 & 1 \\ 5 & 2 & 0 & 8 & 9 & 2 & 1 \\ 6 & 4 & 3 & 8 & 9 & 2 & 1 \end{bmatrix}.$$

d) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Problem 8) (10 points)

a) (4 points) Find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 3 & 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} .$$

b) (2 points) Is there a real eigenbasis of A?

c) (4 points) Find the QR decomposition of the matrix A.

Problem 9) (10 points)

a) (3 points) Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 0 & 7 \\ 0 & 5 & 0 \\ 2 & 0 & 9 \end{bmatrix}$.

b) (4 points) Find the eigenvectors of A.

c) (3 points) Find a matrix S such that $SB = AS$, where B is diagonal.