Name:

MWF10 Oliver knill
MWF11 Anand Patel

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications needed

1) [T] [F] If A is invertible, then A^3 is invertible.

Solution:
$A^{-3} = (A^{-1})^3$ gives the inverse. One could also see it by looking at the kernel: if $A^3\vec{v} = \vec{0}$ with a nonzero $\vec{v}$, then $A(A^2(\vec{v})) = A\vec{u} = \vec{0}$ and A had a nontrivial kernel and could not be invertible.

2) [T] [F] If A^2 is invertible, then A is invertible.

Solution:
If A^2 is invertible then A^2 has full rank, but then also A has full rank.

3) [T] [F] There is a matrix A and a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution and $\ker(A) = \{\vec{0}\}$.

Solution:
Take $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. and $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Now $A\vec{x} = \vec{b}$ has no solution but $\ker(A) = \{\vec{0}\}$.

4) [T] [F] If A is a 3×3 matrix which is a rotation around a line in $\mathbb{R}^3$ then the columns of A form a basis of $\mathbb{R}^3$.

Solution:
The map is invertible and therefore, the columns form a basis.

5) [T] [F] The rank of a diagonal matrix A equals the number of non-zero entries in A.

Solution:
Every nonzero diagonal element will produce a leading 1.

6) [T] [F] Row reduction produces a diagonal matrix which has leading 1 at the places where the original entries were not zero.

Solution:
Start with a 2×2 matrix with 1 everywhere.

7) T **F** $A = \begin{bmatrix} 4/13 & 6/13 \\ 7/13 & 9/13 \end{bmatrix}$ is a projection onto the line spanned by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Solution:
The columns are linearly independent.

8) **T** F The composition of a shear and a rotation in the plane is invertible.

Solution:
Each of these transformations are invertible. So is their composition.

9) T **F** The linear space of cubic polynomials $ax^3 + bx^2 + cx + d$ has dimension 3.

Solution:
$x^3, x^2, x, 1$ form a basis. The dimension is 4.

10) **T** F If A is a 5×5 matrix such that $A^2 = I_5$, (where I_5 is the identity matrix), then A is invertible.

Solution:
If A were not invertible, it had a nontrivial $\vec{v}$ in the kernel and $A\vec{v} = \vec{0}$ implies $A^2\vec{v} = \vec{0}$ which would mean A^2 were not invertible.

11) T **F** If A is a 2×2 matrix such that $A^3 = I_2$, then A is the identity matrix I_n.

Solution:
A could be a rotation by $2\pi/3$ for example.

12) T **F** There exists an invertible 3×3 matrix A such that 7 of its entries are 0.

Solution:
There can only be 2 nonzero entries. We need at least 3 to have an invertible matrix.

13) | T | F | It can happen that in rref(A) there are rows for which all entries are nonzero.

Solution:
Yes, $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is an example. Note that this is an example where all entries of the matrix are nonzero, even so the matrix is in rref. For $m \times n$ matrices with $m > 1$, this is not possible.

14) | T | F | It is possible that rref(A) has columns for which all entries are nonzero.

Solution:
Yes, $A = \begin{bmatrix} 1 \end{bmatrix}$ is an example. An other example is $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

15) | T | F | The rank of a 3×4 matrix may be 4.

Solution:
Look at the row reduced echelon form. There can be maximally 3 leading 1.

16) | T | F | The rank of a 4×3 matrix may be 4.

Solution:
Again, look at the row reduced echelon form. There can be maximally 3 leading 1.

17) | T | F | There is an invertible 2×2 matrix S such that $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Solution:
Just plug in a general matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and check $\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} S = S \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$ which implies $a = b = c = d = 0$. One could also see it differenty: the first matrix satisfies $A^2 = 0$, the second matrix not. This would contradict $S^{-1}A^2S = (S^{-1}AS)(S^{-1}AS) = B^2$.

18) | T | F | The number of leading 1 entries in $\mathrm{rref}(A)$ is the dimension of the image of A.

Solution:
The rank is indeed dimension of the image.

19) | T | F | If A and B are invertible $n \times n$ matrices, then so is $A - B$.

Solution:
False, take for example $B = -A$ where A is invertible.

20) | T | F | There exists a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ for which $\ker(T) = \mathrm{im}(T)$.

Solution:
If true, the dimensions of the image and the kernel would be the same integer k. The rank-nulliety theorem would show that $3 = \dim(\ker(A) + \dim(\mathrm{im}(A) = k + k$ is an even number.

[ ] Total

Problem 2) (10 points)

Which of the following matrices are in reduced row-echelon form? We do not need explanations in this question.

Matrix	IS	IS NOT	Matrix	IS	IS NOT
a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$			b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
c) $\begin{bmatrix} 1 & 6 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$			d) $\begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
e) $\begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$			f) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$		
g) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$			h) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
i) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$			j) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$		

Solution:

a) no
b) no
c) yes
d) no
e) yes
f) no
g) yes
h) yes
i) no
j) no

Problem 3) (10 points)

In problems a)-b), you have to find all the solutions using Gauss-Jordan elimination.

a) (4 points) Find all solutions of $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 5 \\ 2 & 3 & 4 \end{bmatrix}.$$

Solution:
$x = -5, y = 5, z = -1.$

b) (4 points) Find all solutions to the following system of linear equations

$$\begin{aligned} x + y + z + w &= 4 \\ x - y + z - w &= 2 \\ 2x + 2y + 2z + 2w &= 8 \end{aligned}$$

c) (2 points) You have a solution $x \in \mathbb{R}^9$ of a system of linear equations $Ax = b$, where A is a 7×9 matrix, and b is a given vector in $\mathbb{R}^7$. Is it possible to find an other solution $Ay = b$, where y is different from x?

Solution:
To find a solution, we row reduce the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & -1 & 2 \\ 2 & 2 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We could have added an other trivial row and reduced

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & -1 & 2 \\ 2 & 2 & 2 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In any case, we see that $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a special solution of the system. We can add elements of the kernel of the coefficient matrix A and still have solutions. So, lets compute the kernel. We know the row reduction of A already $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There are two redundant columns for which we introduce free variables: $w = t, z = s, x = -t, y = -s$ so that a general element in the kernel is

$$s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

The general solution to the problem is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

c) The matrix has more columns than rows. So there are free variables. The kernel has at least dimension 2.

Problem 4) (10 points)

Which of the following sets are linear subspaces of some $\mathbb{R}^n$? You do have to give explanations.

1) (2 points) The image of the transformation $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

2) (2 points) The kernel of the projection from $\mathbb{R}^3$ onto the xy-plane.

3) (2 points) The solutions of the equation $2x + 3y - 5z = 12$ in $\mathbb{R}^3$.

4) (2 points) All the vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ which satisfy $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$.

5) (2 points) All the points $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{x}\}$, where A is a given $n \times n$ matrix.

Solution:
a) The image of a linear map is always a linear subspace of the codomain. In this case, it is $\mathbb{R}^2$.
b) The kernel of a linear map is always a linear subspace of the domain.
c) The solution space does not contain the vector $\vec{0}$, so it can't be a linear subspace.
d) The solution is the set $\{x^2 + y^2 + z^2 = 0\}$ which is the trivial subspace $\{0\}$ of $\mathbb{R}^3$.
e) It is the kernel of the matrix $B = A - I_2$. We know that the kernel of a matrix is a linear subspace.

Problem 5) (10 points)

Let A be a shear in the plane along the x-axis which maps $\vec{e}_1$ to $\vec{e}_1$ and sends $\vec{e}_2$ to $2\vec{e}_1 + \vec{e}_2$. One calls this transformation also a horizontal shear. Let B the the projection onto the x-axis.

a) (4 points) Find the matrices A, B, AB and BA.

b) (3 points) What is A^{100}?

c) (3 points) What is $(AB)^{10}$?

Solution:
a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. $BA = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.
b) $A = \begin{bmatrix} 1 & 200 \\ 0 & 1 \end{bmatrix}$.
c) Because $(AB)^2 = AB$ and $(BA)^2 = BA$, we also have $(AB)^{10} = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})^{10} = AB =$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 6) (10 points)

a) (3 points) Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is a reflection over the x-axis. Find the matrix of T.

b) (4 points) Let S be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ obtained by first reflecting over the x-axis, then reflecting over the y-axis, and finally reflecting over the line $y = x$. Find the matrix A of S.

c) (3 points) Find the inverse of A, where A is the matrix you found in part (b).

Solution:

a) $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so the matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

b) First let's find $S(\vec{e}_1)$. Reflecting over the x-axes sends e_1 to $\vec{e_1}$. Reflecting over the y-axes sends e_1 to $-\vec{e_1}$. Thus $S(\vec{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Now, we'll find $S(\vec{e}_2)$. Reflecting over the x-axes sends $\vec{e}_2$ to $-\vec{e}_2$. Reflecting over the line $y = x$ sense e_2 to $-e_1$. Thus $S(e_2) = -e_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Therefore, the matrix of S is $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

c) To find A^{-1}, we row reduce the augmented matrix $[A|I_2]$ to obtain obtain $A^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. (The matrix A defines a reflection about the axis $x = -y$. The inverse is the same.)

Problem 7) (10 points)

a) (3 points) Find a basis for the kernel of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

b) (3 points) Find a basis for the image of the following 4×6 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}.$$

What is the dimension of the image and the dimension of the kernel?

c) (4 points) Find the dimension of the image and kernel of the following 4×100 matrix

$$A = \begin{bmatrix} 1 & 2 & 3\cdots & 99 & 100 \\ 2 & 3 & 4\cdots & 100 & 101 \\ 3 & 4 & 5\cdots & 101 & 102 \\ 4 & 5 & 6\cdots & 102 & 103 \end{bmatrix}.$$

Solution:

a) $\ker(A)$ is the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ satisfying $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0$. The only leading variable is x_1. We may set $x_2 = r, x_3 = s, x_4 = t, x_5 = u$ for free variables r, s, t, u. Then,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r - 3x - 4t - 5u \\ r \\ s \\ t \\ u \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of $\ker(A)$.

Solution:

b) The image of A is spanned by the columns of A. All but the first column are redundant. A basis of the image is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Thus, $\dim(\mathrm{im}(A) = 1$. By the rank-nullity theorem, $\dim(\ker(A)) = 6 - \dim(\mathrm{im}(A)) = 5$.

Solution:

c) The image of A is spanned by the columns of A. Let $\vec{v}_1, ..., \vec{v}_{100}$ be the columns of A. The vectors $\vec{v}_1$ and $\vec{v}_2$ are certainly linearly independent since they are not scalar multiple of each other. If $n \geq 2$, then

$$\vec{v}_{n+1} - \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = n(\vec{v}_2 - \vec{v}_1)$$

so that $\dim(\text{im}(A) = 2$. By the rank-nullity theorem, $\dim(\ker(A) = 100 - \dim(\text{im}(A)) = 98$.

Problem 8) (10 points)

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $\mathcal{B}$ be the basis $\{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$.

a) (3 points) Find the coordinates of $\vec{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ in the basis $\mathcal{B}$.
In other words, find the $\mathcal{B}$-coordinates of $\vec{v}$.

b) (4 points) The matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ defines a linear transformation $T(\vec{x}) = A\vec{x}$. What is the $\mathcal{B}$-matrix of T?

c) (3 points) Is there a different basis $\mathcal{B}$ such that the $\mathcal{B}$-matrix of T is $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$? Explain briefly. (Here, T is the linear transformation defined in part (b).)

Solution:

The coordinate transformation is given by the matrix $S = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ which contains the basis vectors in the columns. Its inverse is $S^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. The new coordinates are $[\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v} = \boxed{\begin{bmatrix} 5 \\ -4 \end{bmatrix}}$.

b) The matrix in the new coordinates is $B = S^{-1}AS = \boxed{\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}}$.

c) $\boxed{\text{No}}$. The matrix $C = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$ is not invertible. If $S^{-1}AS = C$, then $AS = SC$. This is not possible because the left hand side an invertible matrix AS and on the right hand side a noninvertible matrix CS.

Problem 9) (10 points)

You are given a matrix A which is a 5×6 matrix of rank 5. In each of the following questions, we need not only the answer but also a short explanation.

a) (2 points) Can the matrix A be invertible ?

b) (3 points) What is the dimension of the image?

c) (3 points) What is the dimension of the kernel?

d) (2 points) How many solutions will the equation $A\vec{x} = \vec{b}$ have?

Solution:

a) $\boxed{\text{No}}$, the matrix is not a square matrix.

b) The dimension of the the image is the rank of A which is $\boxed{5}$.

c) The kernel is $\boxed{\text{1-dimensional}}$ by the rank-nullity theorem.

d) We can add nonzero kernel elements $\vec{v}$ to a solution $\vec{x}$ so that with $A\vec{x} = \vec{b}$ we still have a solution $A(\vec{x} + \vec{b}) = \vec{b}$. This shows that we either have infinitely many or no solution at all. Because the image is the entire codomain, for every $\vec{b} \in \mathbb{R}^5$, there is $\vec{x}$ in $\mathbb{R}^6$ such that $A\vec{x} = \vec{b}$. We always have a solution. Therefore, there must be $\boxed{\text{infinitely many solutions}}$.

Problem 10) (10 points)

A general shear is a linear transformation T for which there is a vector $\vec{v}$ with $T(v) = v$ and such that for all vectors $T(x) - x$ is parallel to v. Is the linear transformation $T(x) = A(x)$ with

$$A = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

a general shear? If so, find the line along which it is centered. It is enough to give a nonzero vector $\vec{v}$ in that line.

Solution:

Check that $T(\vec{x}) - \vec{x}$ is parallel to a fixed vector. It is enough to check that for two vectors: form $T\vec{e}_1 - \vec{e}_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ and $T\vec{e}_2 - \vec{e}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Because both vectors are parallel to $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we know also that linear combinations of $\vec{e}_1$ and $\vec{e}_2$ are parallel to $\vec{v}$. The transformation is a shear.