

Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or un-staple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

- 1)  T  F     The rank of  $A^{-1}$  is always equal to the rank of  $A$  if  $A$  is an invertible matrix.

**Solution:**  
It has to have full rank in order that it is invertible.

- 2)  T  F      $\text{rank}(A - B) = \text{rank}(A) - \text{rank}(B)$  for all  $2 \times 2$  matrices.

**Solution:**  
Take  $A = B = I_n$ .

- 3)  T  F     The row reduced echelon form of an invertible  $3 \times 3$  matrices is invertible.

**Solution:**  
It is the identity

- 4)  T  F     The set of cubic polynomials  $ax^3 + bx^2 + cx + d$  is a three dimensional vector space.

**Solution:**  
It is 4-dimensional.

- 5)  T  F     A system of linear equations has either 0, 1 or  $\infty$  many solutions.

**Solution:**  
This is an important property for systems of linear equations.

- 6)  T  F     A reflection in the plane at the  $x$  axes is similar to the reflection at the  $y$  axes.

**Solution:**  
Just take  $S(e_1) = e_2, S(e_2) = e_1$ . This conjugates the two reflections.

- 7)  T  F Every basis of  $\mathbb{R}^3$  contains exactly 3 vectors in it.

**Solution:**

The number of basis vectors is called the dimension. We have seen that this number does not depend on the choice of the basis.

- 8)  T  F A  $4 \times 4$  matrix can have  $\dim(\text{im}(A)) = \dim(\text{ker}(A))$ .

**Solution:**

An example is  $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ .

- 9)  T  F The rank of a  $7 \times 3$  matrix can be 4.

**Solution:**

We can not have more than 3 leading 1's because each leading one is in its own column.

- 10)  T  F If  $\{v_1, v_2, v_3, v_4\}$  is a set of vectors spanning a linear subspace  $V$  of  $\mathbb{R}^9$ , then  $\dim(V) \geq 4$ .

**Solution:**

It can also be smaller. We can for example take  $v_1 = v_2 = v_3 = v_4$  in which case the linear subspace is 1 dimensional.

- 11)  T  F If  $A$  is a  $7 \times 5$  matrix, then the dimension of  $\text{ker}(A)$  is at least 2.

**Solution:**

The nullity can be 0 since we can have a leading 1 in  $\text{rref}(A)$  in every column.

- 12)  T  F The difference  $A - B$  of 2 invertible  $5 \times 5$  matrices  $A, B$  is invertible.

**Solution:**

We can take  $A = I_n$  and  $B = I_n$ . Then the difference is not invertible.

- 13)  T  F If  $A\vec{x} = \vec{0}$  has a nonzero solution, where  $A$  is a  $4 \times 4$  matrix, then  $\text{rank}(A) \leq 3$ .

**Solution:**

The kernel is at least 1 dimensional. The rank nullity theorem implies that the image is maximally 3 dimensional.

- 14)  T  F If  $\vec{b}$  is in  $\text{im}(A)$ , then  $A\vec{x} = \vec{b}$  has exactly one solution.

**Solution:**

There can be a kernel.

- 15)  T  F If  $A$  and  $B$  are  $2 \times 2$  matrices and  $A \cdot B$  is the identity matrix  $I_2$ , then  $A$  and  $B$  are both invertible.

**Solution:**

Yes.

- 16)  T  F If  $\vec{v}$  is a nonzero vector in the kernel of  $A$ , then  $\vec{v}$  is perpendicular to every row vector of  $A$ .

**Solution:**

This is what  $A\vec{v} = \vec{0}$  means.

- 17)  T  F If  $AB = I_2$  for an  $2 \times 3$  matrix  $A$  and  $B$  is a  $3 \times 2$  matrix, then  $BA = I_3$ .

**Solution:**

This is already not true for  $A = [1, 0], B = [1, 0]^T$  where  $AB = I_1$  and  $BA$  is a projection.

- 18)  T  F If a  $2 \times 2$  matrix different from the identity is its own inverse then it is a reflection at a line.

Problem 2) (10 points) No justifications are needed.

**Solution:**

It can be a reflection at a point.

- 19)  T  F The set of vectors  $(x, y)$  in  $R^2$  such that  $|x| + y = 0$  is a linear subspace of  $R^2$ .

**Solution:**

It is not closed under scalar multiplication.

- 20)  T  F The space of all real  $2 \times 3$  matrices is a linear space.

**Solution:**

Check the three properties.

Total

- a) (5 points) Which of the following matrices are in row reduced echelon form?

Matrix	is in row reduced echelon form
$\begin{bmatrix} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	

- b) (5 points) Check the matrices which are invertible:

Matrix	invertible
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	

**Solution:**

- a) The first 2 are in row reduced echelon form.  
 b) All except the third one are invertible.

Problem 3) (10 points) No justifications are necessary.

- a) (3 points) Which of the following matrices either perform a rotation dilation or a reflection dilation? Check the corresponding boxes (it is also possible that both cases are unchecked):

Matrix	rotation dilat.	reflection dilat.	Matrix	rotation dilat.	reflection dilat.
$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$			$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$		
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$			$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$		

**Solution:**

Matrix	rotation dilat.	reflection dilat.	Matrix	rotation dilat.	reflection dilat.
$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	X		$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$		X
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$			$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$		

- b) (2 points) Which of the following sets are linear spaces?

The space of all ...	Check
functions $f(x) = a \sin(x)$	
all $2 \times 2$ rotation matrices	

The space of all ...	Check
$(x, y, z) \in \mathbf{R}^3$ satisfying $2x + y - 4z = 1$	
smooth functions satisfying $f(x) = -f(-x^2)$	

**Solution:**

The space of all ...	Check
functions $f(x) = a \sin(x)$	x
all $2 \times 2$ rotation matrices	

The space of all ...	Check
$(x, y, z) \in \mathbf{R}^3$ satisfying $2x + y - 4z = 1$	
smooth functions satisfying $f(x) = -f(-x^2)$	x

- c) (5 points) Match the matrices with their actions:

A-F	domain	codomain	A-F	domain	codomain

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

**Solution:**

A-F	domain	codomain	A-F	domain	codomain
B			D		
A			E		
F			C		

**Solution:**

a) Row reduction gives

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two leading 1, the first two columns of the original matrix therefore form the basis for the image. The answer is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

b) Row reduction of  $A$  gives

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see two free variables in the last two rows and get  $x - s + 2t = 0, y + 2s + 3t = 0, z = s, w = t$  leading to the kernel

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the kernel is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Problem 4) (10 points)

a) (5 points) Find a basis of the image of the following chess matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

b) (5 points) Find a basis for the linear subspace of all vectors in  $\mathbf{R}^4$  which are perpendicular to the columns of the matrix

$$A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}.$$

Problem 5) (10 points)

a) (5 points) Invert the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

by using row reduction on an augmented  $4 \times 8$  matrix

b) (5 points) Find a basis for the linear space of vectors perpendicular to the kernel of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

**Solution:**

a) To get the inverse, we row reduce the augmented matrix

$$A = \left[ \begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

A good start is to place the last row on the top:

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right].$$

Then subtract the third row from the second row

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right].$$

Finally subtract the last row from the second last

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right].$$

We can now see the inverse:

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

b) To get the kernel of  $A$ , we row reduce  $A$  and get

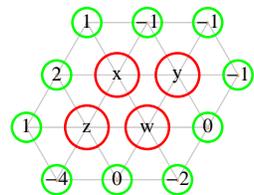
$$\text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

We see one leading 1 and have therefore, one free variable  $t$ . The equations  $x+t=0, y=t$  show that the kernel is the linear space spanned by  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . A basis for the space of vectors perpendicular to the kernel is the kernel of

$$\begin{bmatrix} 1 & -1 \end{bmatrix}.$$

which is spanned by  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

Problem 6) (10 points)



Ruth has a herb garden. The humidity of its soil has the property that at any given point the humidity is the sum of the neighboring humidities. Samples are taken on a hexagonal grid on 14 spots. The humidity at the four locations  $x, y, z, w$  is unknown. The corresponding

equations 
$$\begin{cases} x = y+z+w+2 \\ y = x+w-3 \\ z = x+w-1 \\ w = x+y+z-2 \end{cases}$$
 lead to the following problem:

Find all the solutions to the equations

$$\begin{cases} x - y - z - w = 2 \\ -x + y - w = -3 \\ -x + z - w = -1 \\ -x - y - z + w = -2 \end{cases}$$

using row reduction.

**Solution:**

We row reduce the augmented matrix

$$B = \left[ \begin{array}{cccc|c} 1 & -1 & -1 & -1 & 2 \\ -1 & 1 & 0 & -1 & -3 \\ -1 & 0 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -2 \end{array} \right]$$

and get after a few row reduction steps (left out here)

$$\text{rref}(B) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right].$$

There is exactly one solution.  $x = 2, y = -1, z = 1, w = 0$ .

Problem 7) (10 points)

a) (6 points) Find a basis of the space  $V$  of all vectors perpendicular to the three vectors

$$\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

b) (4 points) Use a) to find a basis for  $R^4$  which contains  $v_1, v_2, v_3$ .

**Solution:**

a) Define

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

We have to find the kernel of  $A$ . The row reduction of  $A$  is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Problem 8) (10 points)

a) (5 points) The projection-dilation matrix  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$  in the basis

$$\mathcal{B} = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is given by a matrix  $B$ . Find this  $3 \times 3$  matrix  $B$ .

b) (5 points) A linear transformation  $T$  satisfies

$$T(v_1) = v_2, T(v_2) = v_3, T(v_3) = v_1$$

where  $v_1, v_2, v_3$  are given in a). Find the matrix  $R$  implementing this transformation in the standard basis.

**Solution:**

a)

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B = S^{-1}AS = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

b)  $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}.$

Problem 9) (10 points)

a) (5 points) Find  $A^{10}$  where  $A = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ .

b) (5 points) Find a  $2 \times 2$  matrix  $X$  satisfying  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} X \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

**Solution:**

a) You might spot the reflection dilation  $B = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$  and a projection dilation  $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . We have  $B^2 = 25I_2$  and  $C^2 = 2C$ . This gives  $B^{10} = 25^5 I_2$  and  $C^{10} = 2^9 C$ .  
Therefore,

$$A^{10} = \begin{bmatrix} 25^5 & 0 & 0 & 0 \\ 0 & 25^5 & 0 & 0 \\ 0 & 0 & 2^9 & 2^9 \\ 0 & 0 & 2^9 & 2^9 \end{bmatrix}$$

b)  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} X \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -4 & 5 \end{bmatrix}$ .

Now multiply from the right with  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$  and from the left with  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1}$  to isolate

$$X = \begin{bmatrix} 1/2 & 2 \\ -5/3 & 1 \end{bmatrix}.$$