DIFFERENTIAL OPERATORS Math 21b, 2010

The handout contains the homework for Friday November 19, 2010. The topic are linear transformation on the linear space $X = C^\infty$ of smooth functions. Remember that a function is called **smooth**, if we can differentiate it arbitrarily many times.

Examples: $f(x) = \sin(x) + x^2$ is an element in C^∞. The function $f(x) = |x|^{5/2}$ is not in X since its third derivative is no more defined at 0. The constant function $f(x) = 2$ is in X.

X is a linear space because it contains the zero function and if f, g are in X then $f + g, \lambda f$ are in X. All the concepts introduced for vectors can be used for functions. The terminology can shift. An eigenvector is also called **eigenfunction**.

A map $T : C^\infty \to C^\infty$ is called a **linear operator** on X if the following three conditions are are satisfied:

(i) $T(f + g) = T(f) + T(g)$
(ii) $T(\lambda f) = \lambda T(f)$
(iii) $T(0) = 0$.

An important example of a linear operator is the differentiation operator D. If p is a polynomial, we can form $p(D)$. For example, for $p(x) = x^2 + 3x - 2$ we obtain $p(D) = D^2 + 3D + 2$ and get $p(D)f = f'' + 3f' + 2f$.

$D(f) = f'$
$p(D)$ **differential operator**

Problem 1) Which of the following maps are linear operators?
a) $T(f)(x) = x^2 f(x - 4)$
b) $T(f)(x) = f'(x)^2$
c) $T = D^2 + D + 1$ meaning $T(f)(x) = f''(x) + f'(x) + f(x)$.
d) $T(f)(x) = e^x \int_0^x e^{-t} f(t)\, dt$.

Problem 2) a) What is the kernel and image of the linear operators $T = D + 3$ and $D - 2$? Use this to find the kernel of $p(D)$ for $p(x) = x^2 + x - 6$?
b) Verify whether the function $f(x) = xe^{-x^2/2}$ is in the kernel of the differential operator $T = D + x$.

Problem 3) In quantum mechanics, the operator $P = iD$ is called the **momentum operator** and the operator $Qf(x) = xf(x)$ is the **position operator**.
a) Verify that every λ is an eigenvalue of P. What is the eigenfunction?
b) What operator is $[Q, P] = QP - PQ$?

Problem 4) The differential equation $f' - 3f = \sin(x)$ can be written as

$$Tf = g$$

with $T = D - 3$ and $g = \sin$. We need to invert the operator T. Verify that

$$Hg = e^{3x} \int_0^x e^{-3t} g(t)\, dt$$

is an inverse of T. In other words, show that the function $f = Hg$ satisfies $Tf = g$.

Problem 5) The operator

$$Tf(x) = -f''(x) + x^2 f(x)$$

is called the **energy operator** of the **quantum harmonic oscillator**.
a) Check that $f(x) = e^{-x^2/2}$ is an eigenfunction of T. What is the eigenvalue?
b) Verify that $f(x) = xe^{-x^2/2}$ is an eigenfunction of T. What is the eigenvalue?