**FINAL EXAM, December 17, 2010** **Math 21b, Fall 2010**

| Name: |
|---|
| |

| |
|---|
| MWF10 Oliver Knill |
| MWF11 Anand Patel |

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, **give details**. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| 11 | | 10 |
| 12 | | 10 |
| 13 | | 10 |
| Total: | | 140 |

Problem 1) (20 points) True or False? No justifications are needed.

1) T F The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is invertible.

**Solution:**
Direct check.

2) T F The difference of two eigenvectors $v_1, v_2$ of a matrix $A$ is always an eigenvector of $A$.

**Solution:**
Only if the two eigenvectors belong to the same eigenvalue, this is true.

3) T F For any invertible $7 \times 7$ matrix $A$, the column space of $A$ is an eigenspace of $A$.

**Solution:**
This is already false for the shear.

4) T F A $2 \times 2$ matrix $A$ is diagonalizable then the matrix $A^2$ is diagonalizable.

**Solution:**
It has the same eigenvectors.

5) T F The function $f(x) = x \sin(x)$ has a Fourier expansion which is a cos series.

**Solution:**
The function is even.

6) T F The space of smooth functions satisfying the equation $f(x) = f(1 + f(x)))$ form a linear space.

**Solution:**
The constant function $f(x) = 1$ and the function $g(x) = x - 1$ are in this space, but the sum $x$ not.

7) T F The sum of all geometric multiplies of a $5 \times 5$ matrix is smaller or equal than the sum of all algebraic multiplicities.

**Solution:**
Each geometric multiplicity is smaller or equal than the algebraic multiplicity

8) | T | **F** | A sum of a $2\times 2$ rotation matrix and a $2\times 2$ reflection matrix is an orthogonal matrix.

**Solution:**
It is the product, not the sum which is an orthogonal matrix.

9) | T | **F** | A discrete dynamical system $x(t+1) = Ax(t)$ with a $4\times 4$ matrix for which all eigenvalues of $A$ are negative, is asymptotically stable.

**Solution:**
It is true for continuous dynamical systems.

10) | **T** | F | There is a matrix $A$ such that $x'(t) = Ax(t)$ and $x(t+1) = Ax(t)$ are both asymptotically stable dynamical systems.

**Solution:**
The eigenvalues have to be between $-1$ and $0$

11) | T | **F** | An orthogonal rotation in the plane composed with an orthogonal projection in the plane is always an orthogonal projection in the plane.

**Solution:**
Rotate by 90 degrees, then project on the $x$ axes.

12) | T | **F** | The function $f(x,t) = e^{-t}\sin(t) - e^{-25t}\sin(5t)$ satisfies the heat equation $f_t = f_{xx}$

**Solution:**
This appears to be the solution formula but there are $\sin(nt)$ terms instead of $\sin(nx)$ terms.

13) | T | **F** | For a $20\times 10$ matrix $A$, the equation $Ax = b$ has either zero or infinitely many solutions.

**Solution:**
It can have exactly one solution, if we are lucky.

14) T **F** The Jacobian matrix at a equilibrium point $(0,0)$ of a nonlinear system $x' = x + y^2$, $y' = x - y^2$ is invertible.

**Solution:**
The eigenvalues are purely imaginary in this case and do not have negative real part.

15) **T** F There are matrices which are diagonalizable over the complex numbers but not over the reals.

**Solution:**
A rotation by 30 degrees is a concrete example.

16) T **F** $||\sin(5x) - \sin(10x)|| = 2$, where the length $||f||$ of $f$ is defined by the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\,dx$.

**Solution:**
This is Parseval's equality. But the result is $\sqrt{2}$

17) T **F** The sum of two eigenvalues of a non-invertible $3\times3$ matrix $A$ is an eigenvalue of $A$.

**Solution:**
Take a matrix with eigenvalues $0, 1$.

18) T **F** For the differential equation $x''(t)+x(t) = \sin(t)$, all solutions are bounded.

**Solution:**
This is a resonance case.

19) **T** F If a $2 \times 2$ matrix $A$ is similar to a $2 \times 2$ matrix $B$, then the transposed matrix $A^T$ is similar to $B^T$.

**Solution:**
Just take the transpose of $B = S^{-1}AS$ to get $B^T = S^T A^T (S^T)^{-1}$ showing that $B^T$ is similar to $A^T$.

20) T **F** A $3 \times 3$ matrix $A$ is invertible if and only if all its eigenvalues are positive.

**Solution:**
They can also be negative.

Problem 2) (10 points) no justifications needed

a) (5 points) Draw with your pen a connection between any pair of matrices which have the same eigenvalues and which are also similar.

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

b) (5 points) Match the initial value problem with their solutions. Each function matches exactly one of the differential equations. The initial condition is $f(0) = 1$ $f'(0) = 0$ in all cases.

| enter A-D | initial value problem |
|---|---|
| | $f''(t) + f(t) = 2\sin(t)$ |
| | $f''(t) + f(t) = 6\sin(2t)$ |
| | $f''(t) - f(t) = 2\sin(t)$ |
| | $f''(t) - f(t) = 6\sin(2t)$ |

| label | solution |
|---|---|
| A) | $f(t) = \cos(t)_{+}\sin(t) - t\cos(t)$ |
| B) | $f(t) = -0.7e^{-t} + 1.7e^{t} - 1.2\sin(2t)$ |
| C) | $f(t) = e^{t} - \sin(t)$ |
| D) | $f(t) = \cos(t) + 4\sin(t) - 2\sin(2t)$ |

**Solution:**
a) $B, A, E$ on the right hand side are connected. Additionally, $C, D$ on the left hand side are connected.
b) A,D,C,B.

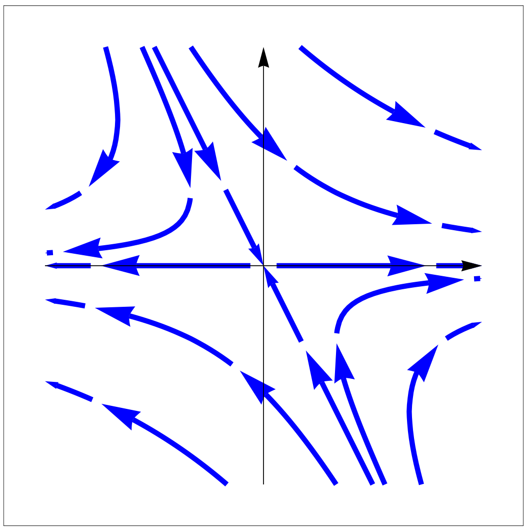
Problem 3) (10 points) no justifications needed

a) (5 points) Match the discrete dynamical systems $x(t+1) = Ax(t)$ with the phase portraits. Enter a)-d).

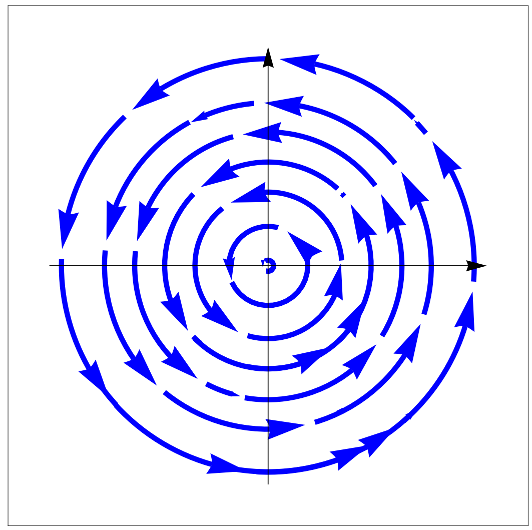
| matrix | $x(t+1) = Ax$ |
|---|---|
| $A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$ | |
| $A = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}$ | |
| $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ | |
| $A = \begin{bmatrix} 2 & -1 \\ 0 & 1/2 \end{bmatrix}$ | |

b) (5 points) Match the continuous dynamical systems $x' = Ax(t)$ with the phase portraits. Enter a)-d).

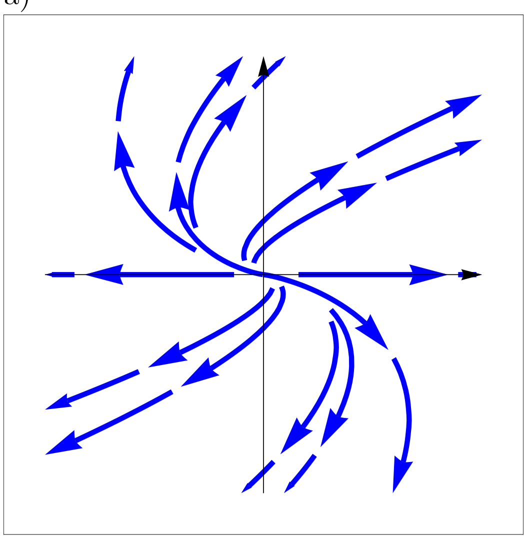
| matrix | $\frac{d}{dt}x(t) = Ax(t)$ |
|---|---|
| $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ | |
| $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ | |
| $A = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}$ | |
| $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ | |



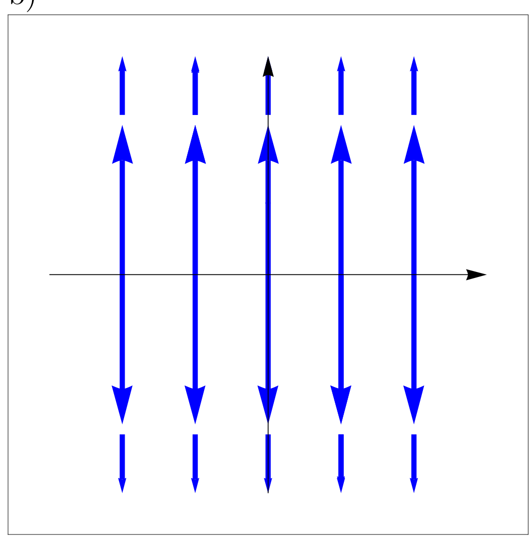
a)



b)



c)



d)

**Solution:**
a) Look at the eigenvalues. In each case they can be read off from the matrix $\boxed{\text{cbda}}$
b) Similarly for discrete systems. But now it is the absolute value of the eigenvalues which matters $\boxed{\text{bdac}}$

| Problem 4) (10 points) |
|---|

We aim to find all the solutions of the following system of linear equations.

$$\left| \begin{array}{ccccccccccccl} x & + & 2y & + & 3z & + & 3u & + & 2v & + & w & = 3 \\ & & y & + & 2z & + & 2u & + & v & & & = 2 \\ & & & & z & + & u & & & & & = 1 \end{array} \right|$$

a) (2 points) Rewrite the system in matrix form $A\vec{x} = \vec{b}$.

b) (5 points) Row reduce the augmented matrix $[A|b]$.

c) (3 points) Write down the general solution.

**Solution:**
The system $Ax = b$ has 6 variables and three equations. a) With the $3 \times 6$ matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

and $b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix}$ the system of linear equations is equivalent to $A\vec{x} = \vec{b}$.

b) The augmented matrix is

$$[A|b] = \begin{bmatrix} 1 & 2 & 3 & 3 & 2 & 1 & | & 3 \\ 1 & 1 & 2 & 2 & 1 & 1 & | & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & | & 1 \end{bmatrix}.$$

We have

$$\text{rref}([A|b]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & | & 1 \end{bmatrix}.$$

c) There are three leading 1 and three free variables lets call them a,b,c

$$\text{rref}([A|b]) = \begin{bmatrix} & & & a & b & c & | & \\ 1 & 0 & 0 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & | & 1 \end{bmatrix}.$$

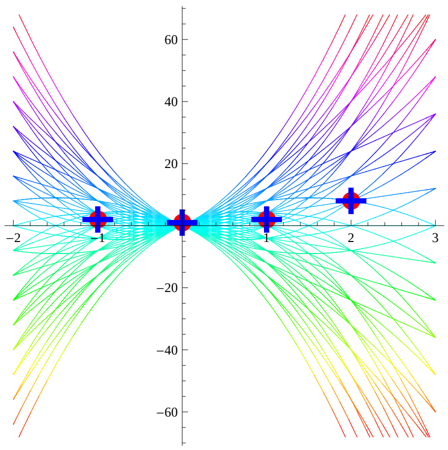
This system can be rewritten as $x + c = 0, y + b = 0, z + a = 1, u = a, v = b, w = c$ leading to the general solution

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 5) (10 points)

Using the least square method, find the best parabolic function of the form $ax + bx^2 = y$ which fits the data points

$$(1,2),(0,1),(2,8),(-1,1)\ .$$




**Solution:**
Write a system $A\vec{x} = \vec{b}$ of equations which tells that all points are on the curve:

$$\begin{aligned} 1+1 &= 2 \\ 0+0 &= 1 \\ 2+4 &= 1 \\ -1+1 &= 2 \end{aligned}$$

The matrix is

$$A = \begin{bmatrix} 1+1 \\ 0+0 \\ 2+4 \\ -1+1 \end{bmatrix} .$$

Now apply the **least square solution formula** $\vec{x} = A(A^TA)^{-1}A^T\vec{b}$ to get the best solution. We get $A^TA = \begin{bmatrix} 6 & 8 \\ 8 & 16 \end{bmatrix}$ which has the inverse $(A^TA)^{-1} = \begin{bmatrix} 18 & -8 \\ 8 & 6 \end{bmatrix}/44$. With $A^Tb = \begin{bmatrix} 16 \\ 35 \end{bmatrix}$ we end up with the solution $\boxed{\vec{x} = (a,b) = (13,37)/22}$.

Problem 6) (10 points)

a) (5 points) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 9 & 9 & 9 \\ 9 & 3 & 9 & 9 \\ 9 & 9 & 3 & 9 \\ 9 & 9 & 9 & 3 \end{bmatrix} .$$

b) (5 points) Find all the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 6 & 1 & 2 & 0 & 0 \\ 0 & 6 & 1 & 2 & 0 \\ 0 & 0 & 6 & 1 & 2 \\ 2 & 0 & 0 & 6 & 1 \\ 1 & 2 & 0 & 0 & 6 \end{bmatrix} .$$

**Hint:** We have $B = 6I_5 + C + 2C^2$ for some other matrix $C$ about which you know a lot already.

**Solution:**
a) Add 6 times the identity to get a matrix with eigenvalues $36, 0, 0, 0$. The eigenvector to the eigenvalue 36 is $[1,1,1,1]^T$. The eigenvectors to the kernel are $[-1,1,0,0]^T, [-1,0,1,0]^T, [-1,0,0,1]^T$. The matrix $A$ itself has the eigenvalues $\boxed{30, -6, -6, -6}$ and the same eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} .$$

b) The circular matrix

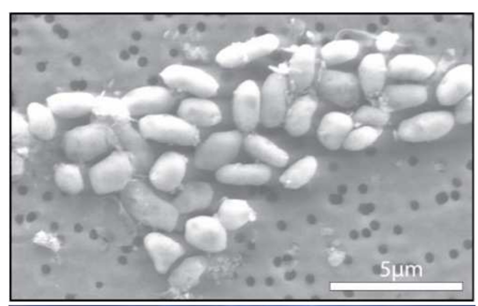
$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

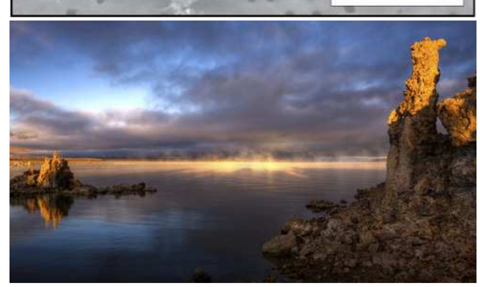
has the eigenvalues $\lambda_k = e^{2\pi i k/5}$ and eigenvectors $[1, \lambda, \lambda^2, \lambda^3, \lambda^4]^T$. The matrix $B$ has the same eigenvectors (there was no need to write this out)

$$\begin{bmatrix} 1 \\ e^{2\pi i1k/5} \\ e^{2\pi i2k/5} \\ e^{2\pi i3k/5} \\ e^{2\pi i4k/5} \end{bmatrix} \begin{bmatrix} 1 \\ e^{2\pi i2k/5} \\ e^{2\pi i4k/5} \\ e^{2\pi i6k/5} \\ e^{2\pi i8k/5} \end{bmatrix} \begin{bmatrix} 1 \\ e^{2\pi i3k/5} \\ e^{2\pi i6k/5} \\ e^{2\pi i9k/5} \\ e^{2\pi i12k/5} \end{bmatrix} \begin{bmatrix} 1 \\ e^{2\pi i4k/5} \\ e^{2\pi i8k/5} \\ e^{2\pi i12k/5} \\ e^{2\pi i16k/5} \end{bmatrix} \begin{bmatrix} 1 \\ e^{2\pi i5k/5} \\ e^{2\pi i10k/5} \\ e^{2\pi i15k/5} \\ e^{2\pi i20k/5} \end{bmatrix} .$$

and the eigenvalues $\boxed{6 + \lambda_k + 2\lambda_k^2}$.

Problem 7) (10 points)



On December 2, 2010, NASA researchers have announced the existence of bacteria for which some phosphor in the DNA is replaced by the element arsenic. Call $p$ the number of phosphor bacteria and $a$ the number of arsenic bacteria. Assume we have initially 100 phosphor bacteria, 0 arsenic bacteria and every night 90 percent of the $p$ bacteria become arsenic bacteria and 80 percent of all arsenic bacteria become phosphorus bacteria. This system is written as $\vec{x}(t+1) = A\vec{x}(t)$ written out as

$$\begin{bmatrix} p(t+1) \\ a(t+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.8 \\ 0.9 & 0.2 \end{bmatrix} \begin{bmatrix} p(t) \\ a(t) \end{bmatrix} .$$

a) (4 points) Find the eigenvalues and eigenvectors of $A$.

b) (4 points) Find a closed form solution of the system.

c) (2 points) The eigenvector $v$ to the largest eigenvalue of $A$ is the final equilibrium distribution. Find it and normalize it so that the sum of the entries is 100. The entries of $v$ now give the final distribution in percentages.

**Solution:**
a) The matrix is a Markov matrix. Its transpose

$$\begin{bmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{bmatrix} .$$

has an eigenvalues 1 because the sum of the row elements add up to 1. The second eigenvalue can be obtained by looking at the trace $\text{tr}(A) = \lambda_1 + \lambda_2 = 0.3$. The second eigenvalue is therefore $\lambda_2 = -0.7$. The eigenvalues of $A$ are also $1, -0.7$ because transposed matrices have the same characteristic polynomial and so eigenvalues. The eigenvector to the eigenvalue 1 is $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$. The eigenvector to the eigenvalue $-0.7$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Write the initial condition as a sum of eigenvectors:

$$\begin{bmatrix} 100 \\ 0 \end{bmatrix} = \frac{100}{17} \begin{bmatrix} 8 \\ 9 \end{bmatrix} - \frac{900}{17} \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

The closed form solution is then

$$\frac{100}{17} e^t \begin{bmatrix} 8 \\ 9 \end{bmatrix} - \frac{900}{17} e^{-0.7t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

c) Normalized, we have to multiply this with $100/(1 + 8/9) = 900/17$ which gives $\boxed{\begin{bmatrix} 800/17 \\ 900/17 \end{bmatrix}}$.

Problem 8) (10 points)

Let $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$.

a) (2 points) Compute $A^2$ and $A^{-1}$.

b) (2 points) Find the $QR$ decomposition of $A$.

c) (2 points) Find the characteristic polynomial $f_A(\lambda)$ of $A$.

d) (2 points) If $A$ is similar to a diagonal matrix $D$, find this matrix $D$, if not tell why there is none.

e) (2 points) Find a $2 \times 2$ matrix $X$ so that $AX = A + A^2$.

a) $A^2 = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix} /14$.
b) The QR factorization is (normalize the first vector to get $u_1$ then subtract $u_1 \cdot v_2$ times $u_1$ and normalize): $u_1 = [4,2]/\sqrt{20}$ and $w_2 = [3,5] - 22/20[4,2] = [-14,28]/10$ to get $[-1,2]/\sqrt{5}$.

$$A = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 2\sqrt{5} & 11/\sqrt{5} \\ 0 & 7/\sqrt{5} \end{bmatrix} .$$

c) $\lambda^2 - 9\lambda + 14$.
d) $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.
e) $X = (1 + A) = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}$ solves it.

## Problem 9) (10 points)

a) (4 points) Find the determinant of the following matrix

$$\begin{bmatrix} 6 & 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 6 & 2 \\ 2 & 2 & 2 & 2 & 6 \end{bmatrix} .$$

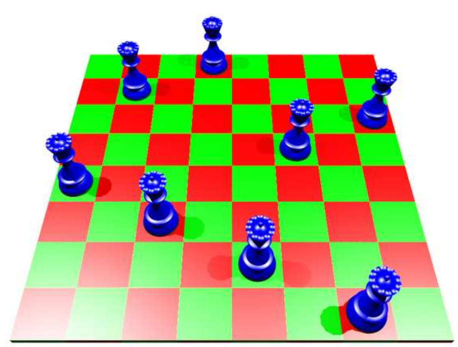
Make sure to mention all tools you need to find the answer.

b) (3 points) Find the determinant of the following matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

Make sure to mention all tools you need to find the answer.

c) (3 points) The following matrix is a solution to the **eight queens puzzle**. Each entry 2 represents a **queen**. No queen can catch any other queen in chess. What is the determinant of this matrix? Make sure to mention all tools you need to find the answer.



$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} .$$

**Solution:**

a) We compute the eigenvalues by subtracting $4I_5$ first. The matrix $B = A - 4I_5$ is easily row reduced and has a 4 dimensional kernel. Because the algebraic multiplicity is larger than equal to 4 and can not be 5, we have 4 eigenvalues 0. The eigenvalues of $B$ being $0, 0, 0, 0, 10$, the matrix $A$ has the eigenvalues $4, 4, 4, 4, 14$. Take the product leading to $14 * 4^4$ to get the determinant. The final answer is $\boxed{3584}$.

b) Row reduce or use the Laplace expansion. We can also see that row 3 minus row 5 gives row 1. The matrix has determinant $\boxed{0}$.

c) There is only 1 pattern. To get the sign, we count the number of up-crossings: $6+4+2+0+2+2+0+0 = 16$. Therefore, the determinant is $(-1)^{16}2^8$ which is $\boxed{256}$.

| Problem 10) (10 points) |
|---|

Solve the following differential equations for which the initial position $f(0)$ and velocity $f'(0)$ is given:

a) (5 points)

$$f''(t) + 9f(t) = e^t, f(0) = 3, f'(0) = 0$$

b) (5 points)

$$f''(t) - 5f'(t) + 6f(t) = 2, f(0) = 3, f'(0) = -4$$

**Solution:**

a) The homogeneous solution is $c_1 \cos(3t) + c_2 \sin(3t)$, where $c_1, c_2$ are constants. The special solution is $e^t/10$. Comparing the initial conditions gives the constants $c_1 = 29/10, c_2 = -1/30$. The solution is

$\boxed{\frac{29}{10}\cos(3t) + \frac{-1}{30}\sin(3t) + \frac{e^t}{10}}$

b) The homogeneous solution is $c_1e^{2t} + c_2e^{3t}$, where $c_1, c_2$ are constants. A special solution is $1/3$. Compare the initial conditions to get the constants $c_1 = 12, c_2 = 28/3$.

$\boxed{12e^{2t} + \frac{28}{3}e^{3t} + \frac{1}{3}}$

| Problem 11) (10 points) |
|---|

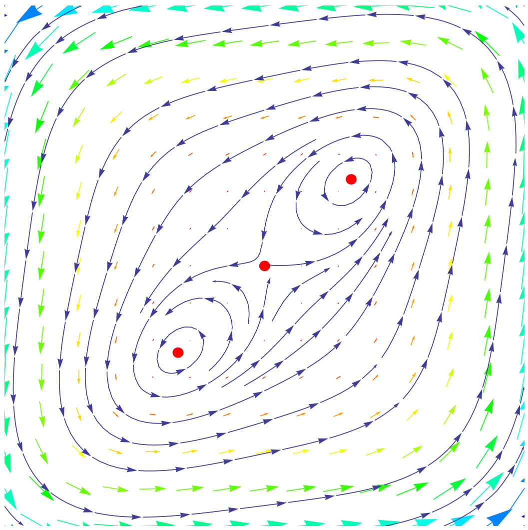
We analyze the following nonlinear dynamical system

$$\begin{aligned} \frac{d}{dt}x &= x - y^3 \\ \frac{d}{dt}y &= x^3 + y \end{aligned}$$

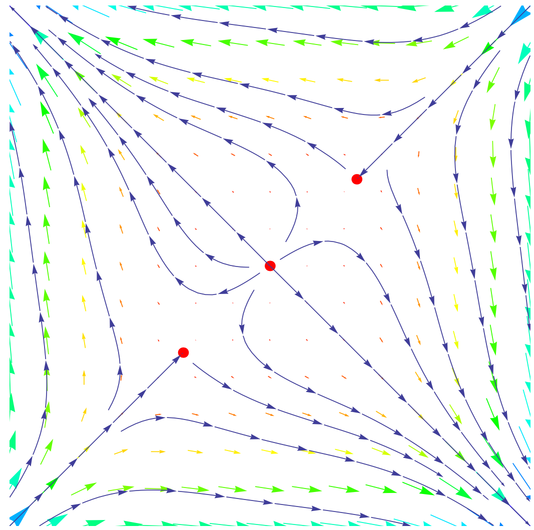
a) (3 points) Find the equations of the null-clines and find all the equilibrium points.

b) (4 points) Analyze the stability of the equilibrium point or equilibrium points.

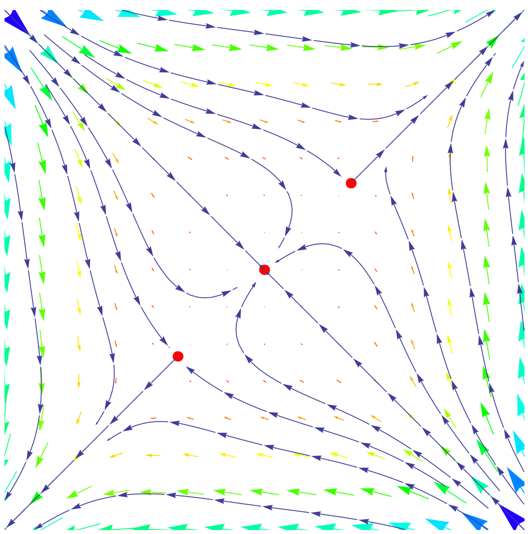
c) (3 points) Which of the four phase portraits A,B,C,D below belongs to the above system? Make sure that also here, you justify your answer, as always.



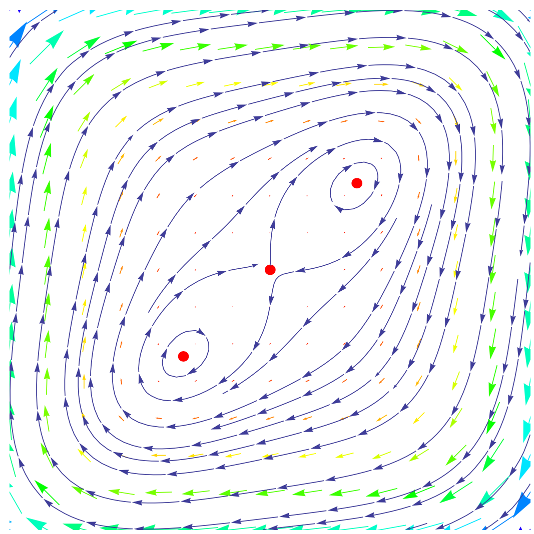
A



B



C

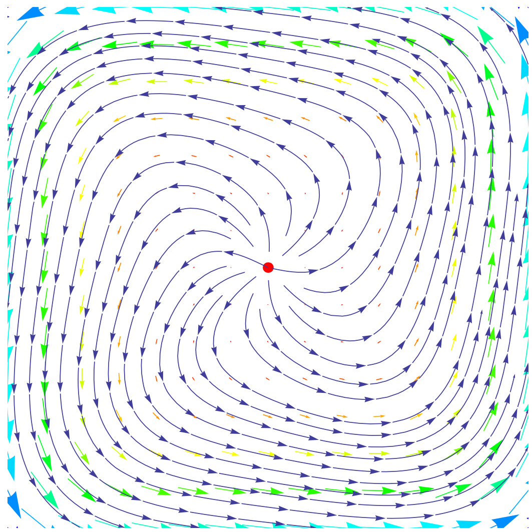


D

**Solution:**
a) The nullclines are the curves $x = y^3$, $y = -x^3$. These cubic curves intersect in one point $(0,0)$ only. There is only one equilibrium point $\boxed{(0,0)}$.

b) The Jacobian matrix at the origin is $\begin{bmatrix} 1 & -3y^2 \\ 3x^2 & 1 \end{bmatrix}$ which is the $\boxed{\text{identity matrix}}$ at the origin. Because it has two eigenvalues 1, the continuous dynamical system is **unstable** there. All solutions close to the origin are pushed away from it.

c) Only the **second phase portrait** has an equilibrium point which is unstable. We counted B as a correct solution therefore. But **None** of the phase portrait matches exactly. Here is the correct phase portrait:

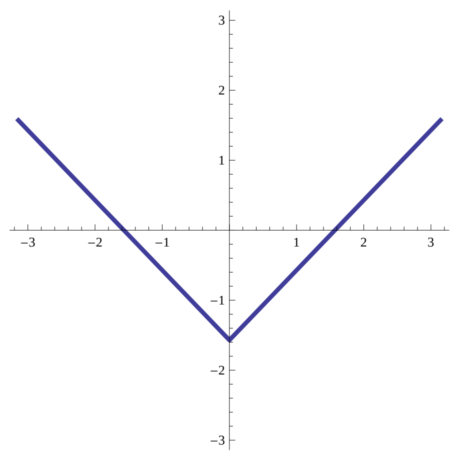


| Problem 12) (10 points) |
|---|

a) (7 points) Find the **Fourier series** of the function

$$f(x) = |x| - \frac{\pi}{2} \, .$$

The graph of the function $f$ on $[-\pi, \pi]$ is displayed to the right.




b) (3 points) Use **Parseval's theorem** to find the value of the integral

$$\sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} (3\sin(5x) + 4\cos(12x) + 3\sin(x) + 4\cos(7x) + 5\cos(x) + 5\sin(2x))^2 \, dx}$$

without evaluating the integral directly.

**Solution:**
a) The function $|x| - \pi/2$ is even so that it has a cos-series. We have $a_0 = (2/\pi)\int_0^\pi (|x| - \pi/2)(1/\sqrt{2}) = 0$. Note that the inner product of $(-\pi/2, \cos(nx)) = 0$ so that we can replace $f$ with $|x|$ and get

$$a_n = \frac{2}{\pi}\int_0^\pi (|x| - \pi/2)\cos(n\pi)\,dx = \frac{2}{\pi}\int_0^\pi x\cos(n\pi)\,dx = \frac{2}{n^2\pi}(\cos(n\pi) - 1)\ .$$

This could be simplified to $\boxed{-4/(n^2\pi) \text{ if } n \text{ is odd and } 0 \text{ else}}$.
b) Only finitely many Fourier coefficients are nonzero. This is the length of a trigonometric polynomial $f(x) = (3\sin(5x) + 4\cos(12x) + 3\sin(x) + 4\cos(7x) + 5\cos(x) + 5\sin(2x))$. The length square is the sum of the squares of the Fourier coefficients. The answer is therefore $\sqrt{9+16+9+16+25+25} = \sqrt{100} = \boxed{10}$.

Problem 13) (10 points)

The partial differential equation

$$u_{tt} = 9u_{xx} + \sin(1000t)$$

describes a **violin string** which is excited with a periodic force from the bow hair. The string is located on $[0, \pi]$ and has the initial condition $u(x, 0) = 2\sin(17x) + 5\sin(10x)$ and initial velocity $u_t(x, 0) = 5\sin(x) - 2\sin(3x)$. Find the motion $u(x, t)$ of the string.

a) (3 points) Find a special solution $u(x, t)$ which does not depend on $x$.

b) (5 points) Find a solution $u(x, t)$ of the homogeneous $u_{tt} = 9u_{xx}$ with the given initial condition.

c) (2 points) Write down the final solution of the problem.

**Solution:**

a) Integrate twice to get $u_p = -\sin(1000t)/1000^2$.

b) With the initial condition $2\sin(17x)$ we get the solution $2\sin(17x)\cos(17ct)$ of the wave equation $u_{tt} = c^2 u_{xx}$. Since $c = 9$, this gives $2\sin(17x)\cos(51t)$. Similarly, with the initial condition $5\sin(10x)$ we get the solution $5\sin(10x)\cos(30t)$. With the initial velocity $5\sin(x)$ we get the solution $5\sin(x)\sin(3t)/3$. With the initial velocity $-2\sin(3x)$ we get the solution $-2\sin(3x)\sin(9t)/9$. The general solution is the sum of these 4 solutions: $u_h(x,t) = 2\sin(17x)\cos(51t) + 5\sin(10x)\cos(30t) + \frac{5}{3}\sin(x)\sin(3t) - \frac{2}{9}\sin(3x)\sin(9t)$.

c) The final solution is the sum $u_p + u_h$ of the solutions found in a) and b).

$$u(x,t) = 2\sin(17x)\cos(51t) + 5\sin(10x)\cos(30t) + \frac{5}{3}\sin(x)\sin(3t) - \frac{2}{9}\sin(3x)\sin(9t) + \frac{-\sin(1000t)}{1000^2}.$$

**Remark.** In this case, no Fourier decomposition was necessary because the functions were already given as a Fourier series. [ If the initial position of the wave would have been $\sum_n b_n \sin(nx)$, and the initial velocity $\sum_n \tilde{b}_n \sin(nx)$ the solution would have been $\sum_n b_n \sin(nx)\cos(3nt) + \sum_n \tilde{b}_n \sin(nx)\sin(3nt)/(3n)$. ]