

Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or un-staple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) True or False? No justifications are needed.

- 1)  T  F If  $A$  is a real  $n \times n$  matrix, then  $A + A^T$  is diagonalizable.
- 2)  T  F The equation  $z^3 = -1$  has three different complex solutions  $z$  and  $z = -1$  is one of them.
- 3)  T  F There is a  $2 \times 2$  projection matrix  $A$  that projects onto a line and a  $2 \times 2$  reflection matrix  $B$  that rotates about a point, so that  $AB$  is a rotation matrix.
- 4)  T  F The transformation  $T(f)(x) = f(f(x))$  is linear on the space  $C^\infty$  of all smooth functions
- 5)  T  F If a continuous function  $f$  defined on the interval  $[-\pi, \pi]$  is both even and odd, then it must be constant function.
- 6)  T  F If  $A$  is a  $2 \times 2$  matrix, then the characteristic polynomial of  $AA^T$  is  $\lambda^2 + c\lambda$  for some real constant  $c$ .
- 7)  T  F The function  $f(t) = e^t$  is an eigenfunction with eigenvalue 1 of the linear operator  $T = D^2$  where  $Df = f'$ .
- 8)  T  F The initial value problem  $f''(x) + f(x) = \sin(x)$ ,  $f''(0) = 1$ ,  $f'(0) = 1$  has exactly one solution.
- 9)  T  F The transformation  $L(A) = A + A^T$  is a linear transformation on  $M_n$  and  $L$  has only the eigenvalues 0 and 2.
- 10)  T  F The set  $X$  of smooth functions  $f(x, y, z)$  which satisfy the  $f_{xx} + f_{yy} + f_{zz} = f$  is a linear space.
- 11)  T  F If  $A$  is a  $5 \times 5$  matrix that has rank 2, then the eigenvalue 0 has algebraic multiplicity 3.
- 12)  T  F Every equilibrium point of a nonlinear system  $\dot{x} = f(x, y)$ ,  $\dot{y} = g(x, y)$  is located on at least one nullcline.
- 13)  T  F The transformation  $T(A) = \text{rank}(A)$  is linear from the space  $M_2$  of all real  $2 \times 2$  matrices to  $\mathbb{R}$ .
- 14)  T  F The  $QR$  decomposition of a  $n \times n$  reflection matrix  $A$  is  $A = QR$ , where  $Q = A$  and  $R = I_n$ .
- 15)  T  F If all eigenvalues of a  $2 \times 2$  matrix  $A$  are zero then  $A$  is the zero matrix.
- 16)  T  F The discrete dynamical system  $x(t+1) = 7x(t) + 3x(t-1)$  has the property that  $|x(t)|$  stays bounded for all initial conditions  $(x(0), x(1))$ .
- 17)  T  F  $\|6\sin(x) + 8\cos(9x)\| = 10$ , where  $\|f\| = \sqrt{\langle f, f \rangle}$  is the length of the function  $f$  with respect to the inner product  $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$ .
- 18)  T  F Any reflection has real eigenvalues and an orthonormal eigenbasis.
- 19)  T  F Given a  $n \times m$  matrix  $A$  with trivial kernel. If  $A(A^T A)^{-1} A^T b = b$  for all  $b$ , then  $n = m$ .
- 20)  T  F If  $A, B$  are  $2 \times 2$  matrices and a system  $Ax = 0$  has infinitely many solutions and the system  $Bx = 0$  has exactly one solution, then  $ABx = 0$  has infinitely many solutions.

Problem 2) (10 points) no justifications needed

a) (5 points) Circle all the matrices which are similar to the matrix  $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ :

$$B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

b) (5 points) Match the differential equations with solution functions. Enter A-E into the boxes to the left. Every choice A-E appears exactly once (despite the fact that two entries on the left appear in pairs):

$$f''(t) + f(t) = t$$

$$f''(t) - f(t) = t$$

$$f''(t) = t$$

$$f''(t) - f(t) = t$$

$$f''(t) + f(t) = t$$

A)  $f(t) = e^t - t$ .

B)  $f(t) = t + \cos(t) - \sin(t)$

C)  $f(t) = 1 - t + t^3/6$

D)  $f(t) = e^t - e^{-t} - t$

E)  $f(t) = t$

Problem 3) (10 points) no justifications needed

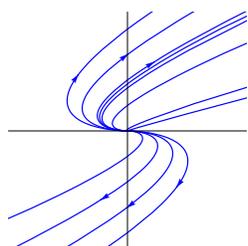
a) (5 points)  $\lambda_1, \lambda_2, \lambda_3$  are the eigenvalues of a  $3 \times 3$  matrix  $A$  which represents a transformation  $T$  in space. Which transformation belongs to these eigenvalues? All except one of the transformations a)-f) appear. Each of the five appears exactly once.

$\lambda_1$	$\lambda_2$	$\lambda_3$	Enter a)-f) here
1	1	1	
-1	-1	1	
0	1	0	
0	1	1	
1	-1	1	

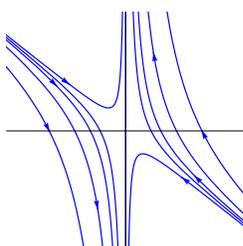
- a) projection onto a plane
- b) reflection at a plane
- c) reflection at a line
- d) projection onto a line
- e) identity matrix
- f) rotation about an axis

b) (5 points) Enter 1)-6) in the first columns and check the boxes. “asymptotically stable” is abbreviated by “stable” and “diagonalizable” means diagonalizable over the complex numbers. The phase portraits belong to the continuous system  $x' = Ax$ . All except one of the phase portrait can be matched. Each of the five appears exactly once.

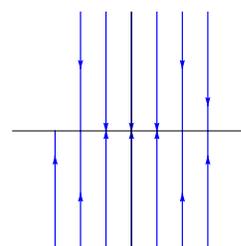
matrix	phase 1)-6)	$\frac{d}{dt}x = Ax$ stable	$x(t+1) = Ax(t)$ stable	$A$ diagonalizable
$\begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix}$				
$\begin{pmatrix} -2 & 0 \\ 4 & 3 \end{pmatrix}$				
$\begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$				
$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$				
$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$				



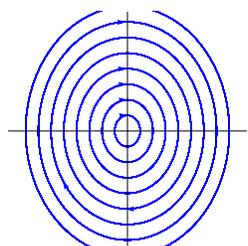
1)



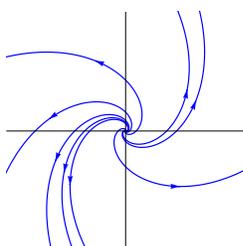
2)



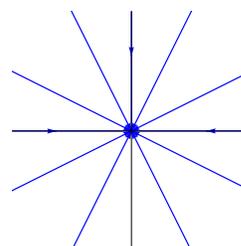
3)



4)



5)



6)

Problem 4) (10 points)

Find all the solutions of the system of linear equations for the 5 variables  $x, y, z, u, v$  using row reduction.

$$\begin{cases} x & + & z & - & u & + & v & = & 2 \\ x & + & y & + & z & + & u & + & v & = & 6 \\ x & + & y & - & z & - & u & + & v & = & 0 \end{cases}$$

Problem 5) (10 points)

a) (7 points) Find the best function  $ax^2 + b2^x = y$  which fits the data points  $(1, 2), (0, 1), (2, 4)$  using the least square method.

b) (3 points) With the data points  $(0, 1), (0, 2), (0, 0)$ , there is more than one minimal solution. Examples are  $x^2 + 2^x = y$  and  $2x^2 + 2^x = y$ . Why does the least square method fail in this case? Choose one of the following explanations. No further explanations are needed in this part b).

The matrix $A$ in the least square solution formula is undefined.	
The matrix $A$ is not invertible.	
The matrix $A^T A$ is not invertible.	

Problem 6) (10 points)

a) (6 points) Find all the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

b) (4 points) Find all the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

Problem 7) (10 points)

a) (8 points) Find a closed form solution of the difference equation

$$x_{n+1} = 2y_n - 3x_n$$

$$y_{n+1} = x_n - 2y_n$$

with initial condition  $x_0 = 5, y_0 = 3$ .

b) (2 points) A system is called **Lyapunov stable** if for all  $(x_0, y_0)$  the sequence  $(x_n, y_n)$  stays in a bounded region  $R(x_0, y_0)$ . (This means that no trajectory can run off to infinity.) Check the following boxes. No further reasoning is needed in this part b).

The system is Lyapunov stable	
The system is asymptotically stable	

Problem 8) (10 points)

On the planet Vulcan, 16 light years from earth, 5 year old Spock is asked to compute the eigenvalues and an orthonormal eigenbasis for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} .$$

He got stuck. Help him.



Problem 9) (10 points)

a) (7 points) Find the determinant of the following matrix

$$\begin{bmatrix} 9 & 1 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 9 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 9 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 1 & 9 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 1 & 1 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

Make sure to mention all tools you need to find the answer.

b) (3 points) Find the determinant of a  $5 \times 5$  matrix  $X$  which satisfies

$$AXA = B ,$$

where  $A$  is a reflection at a one-dimensional line in  $R^5$  and  $B$  is a reflection at a two-dimensional plane in  $R^5$ . As usual, justify your answer.

Problem 10) (10 points)

Find the general solutions for the following differential equations:

a) (3 points)

$$f'' + 16f = \cos(4x), f(0) = f'(0) = 1$$

b) (2 points)

$$f'' = \sin(4x), f(0) = f'(0) = 1$$

c) (3 points)

$$f'' - f' + f = 1, f(0) = 2, f'(0) = 3$$

d) (2 points)

$$f'' - 2f' + f = 0$$

Problem 11) (10 points)

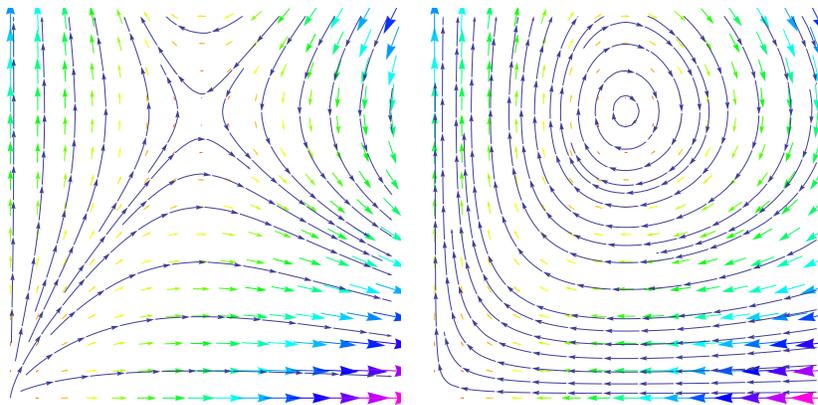
The following nonlinear dynamical system appears as a catalytic network in biology

$$\begin{aligned}\frac{d}{dt}x &= 2x - xy \\ \frac{d}{dt}y &= 3y - xy\end{aligned}$$

a) (3 points) Find the equations of the null clines and find all the equilibrium points.

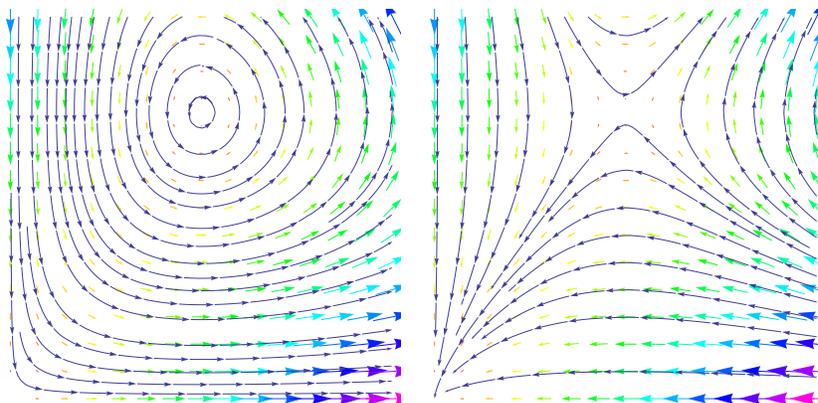
b) (4 points) Analyze the stability of all the equilibrium points.

c) (3 points) Which of the phase portraits A,B,C,D below belongs to the above system?



A

B



C

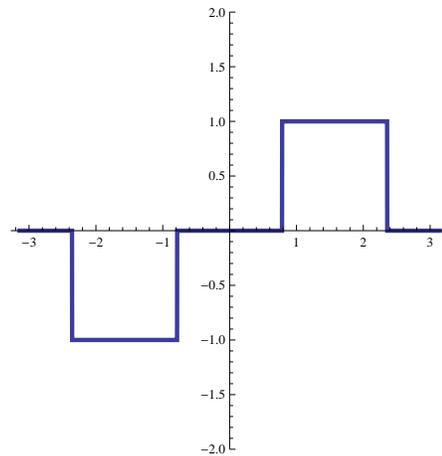
D

Problem 12) (10 points)

a) (7 points) Find the Fourier series of the piecewise constant function

$$f(x) = \begin{cases} 1 & \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \\ -1 & \frac{-3\pi}{4} \leq x \leq \frac{-\pi}{4}. \end{cases}$$

The graph of the function is visible to the right.



b) (3 points) Use Parseval's theorem to find the value of the sum

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

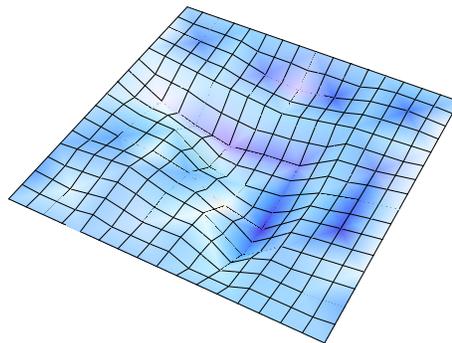
**Hint.** The Fourier coefficients you found in a) are all zero for even  $n$ .

Problem 13) (10 points)

The partial differential equation

$$u_{tt} = T(u) = u_{xx} + u_{yy} - u_{xxxx}$$

is a modification of a two dimensional wave equation and describes a moving membrane  $u(x, y, t)$  with  $-\pi \leq x \leq \pi, -\pi \leq y \leq \pi$ .



a) (2 points) Verify that  $\sin(nx) \sin(my)$  is an eigenvector of the operator  $T$ . Find the corresponding eigenvalue  $\lambda_{nm}$ .

b) (4 points) What is the solution of the above wave type equation if the double sum

$$u(x, y, 0) = \sum_{n,m=1}^{\infty} \frac{1}{nm} \sin(nx) \sin(my)$$

is the initial position of the membrane and the initial velocity of the membrane is zero?

c) (4 points) What is the solution of the above wave type equation if  $u_t(x, y, 0) = \sin(4x) \sin(6y)$  is the initial velocity of the membrane and the initial position  $u(x, y, 0)$  of the membrane is zero everywhere.