

$$x' - \lambda x = 0$$

$$x(t) = Ce^{\lambda t}$$

This first order ODE is by far the most important differential equation. A linear system of differential equation  $x'(t) = Ax(t)$  reduces to this after diagonalization. We can rewrite the differential equation as  $(D - \lambda)x = 0$ . That is  $x$  is in the kernel of  $D - \lambda$ . An other interpretation is that  $x$  is an eigenfunction of  $D$  belonging to the eigenvalue  $\lambda$ . This differential equation describes exponential growth or exponential decay.

$$x'' + k^2 x = 0$$

$$x(t) = C_1 \cos(kt) + C_2 \sin(kt)/k$$

This second order ODE is by far the second most important differential equation. Any linear system of differential equations  $x''(t) = Ax(t)$  reduces to this with diagonalization. We can rewrite the differential equation as  $(D^2 + k^2)x = 0$ . That is  $x$  is in the kernel of  $D^2 + k^2$ . An other interpretation is that  $x$  is an **eigenfunction** of  $D^2$  belonging to the eigenvalue  $-k^2$ . This differential equation describes oscillations or waves.

OPERATOR METHOD. A general method to find solutions to  $p(D)x = g$  is to factor the polynomial  $p(D) = (D - \lambda_1) \cdots (D - \lambda_n)x = g$ , then invert each factor to get

$$x = (D - \lambda_n)^{-1} \cdots (D - \lambda_1)^{-1} g$$

where

$$(D - \lambda)^{-1} g = Ce^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda s} g(s) ds$$

COOKBOOK METHOD. The operator method always works. But it can produce a considerable amount of work. Engineers therefore rely also on cookbook recipes. The solution of an inhomogeneous differential equation  $p(D)x = g$  is found by first finding the **homogeneous solution**  $x_h$  which is the solution to  $p(D)x = 0$ . Then a particular solution  $x_p$  of the system  $p(D)x = g$  found by an educated guess. This method is often much faster but it requires to know the "recipes". Fortunately, it is quite easy: as a rule of thumb: feed in the same class of functions which you see on the right hand side and if the right hand side should contain a function in the kernel of  $p(D)$ , try with a function multiplied by  $t$ . The general solution of the system  $p(D)x = g$  is  $x = x_h + x_p$ .

FINDING THE HOMOGENEOUS SOLUTION.  $p(D) = (D - \lambda_1)(D - \lambda_2) = D^2 + bD + c$ . The next table covers all cases for homogeneous second order differential equations  $x'' + px' + q = 0$ .

$\lambda_1 \neq \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$\lambda_1 = \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$
$ik = \lambda_1 = -\lambda_2$ imaginary	$C_1 \cos(kt) + C_2 \sin(kt)$
$\lambda_1 = a + ik, \lambda_2 = a - ik$	$C_1 e^{at} \cos(kt) + C_2 e^{at} \sin(kt)$

FINDING AN INHOMOGENEOUS SOLUTION. This can be found by applying the operator inversions with  $C = 0$  or by an educated guess. For  $x'' = g(t)$  we just integrate twice, otherwise, check with the following table:

$g(t) = a$ constant	$x(t) = A$ constant
$g(t) = at + b$	$x(t) = At + B$
$g(t) = at^2 + bt + c$	$x(t) = At^2 + Bt + C$
$g(t) = a \cos(bt)$	$x(t) = A \cos(bt) + B \sin(bt)$
$g(t) = a \sin(bt)$	$x(t) = A \cos(bt) + B \sin(bt)$
$g(t) = a \cos(bt)$ with $p(D)g = 0$	$x(t) = At \cos(bt) + Bt \sin(bt)$
$g(t) = a \sin(bt)$ with $p(D)g = 0$	$x(t) = At \cos(bt) + Bt \sin(bt)$
$g(t) = ae^{bt}$	$x(t) = Ae^{bt}$
$g(t) = ae^{bt}$ with $p(D)g = 0$	$x(t) = Ate^{bt}$
$g(t) = q(t)$ polynomial	$x(t) =$ polynomial of same degree

EXAMPLE 1:  $f'' = \cos(5x)$

This is of the form  $D^2f = g$  and can be solved by inverting  $D$  which is integration: integrate a first time to get  $Df = C_1 + \sin(5x)/5$ . Integrate a second time to get

$$f = C_2 + C_1t - \cos(5t)/25$$

This is the operator method in the case  $\lambda = 0$ .

EXAMPLE 2:  $f' - 2f = 2t^2 - 1$

This homogeneous differential equation  $f' - 5f = 0$  is hardwired to our brain. We know its solution is  $Ce^{2t}$ . To get a homogeneous solution, try  $f(t) = At^2 + Bt + C$ . We have to compare coefficients of  $f' - 2f = -2At^2 + (2A - 2B)t + B - 2C = 2t^2 - 1$ . We see that  $A = -1, B = -1, C = 0$ . The special solution is  $-t^2 - t$ . The complete solution is

$$f = -t^2 - t + Ce^{2t}$$

EXAMPLE 3:  $f' - 2f = e^{2t}$

In this case, the right hand side is in the kernel of the operator  $T = D - 2$  in equation  $T(f) = g$ . The homogeneous solution is the same as in example 2, to find the inhomogeneous solution, try  $f(t) = Ate^{2t}$ . We get  $f' - 2f = Ae^{2t}$  so that  $A = 1$ . The complete solution is

$$f = te^{2t} + Ce^{2t}$$

EXAMPLE 4:  $f'' - 4f = e^t$

To find the solution of the homogeneous equation  $(D^2 - 4)f = 0$ , we factor  $(D - 2)(D + 2)f = 0$  and add solutions of  $(D - 2)f = 0$  and  $(D + 2)f = 0$  which gives  $C_1e^{2t} + C_2e^{-2t}$ . To get a special solution, we try  $Ae^t$  and get from  $f'' - 4f = e^t$  that  $A = -1/3$ . The complete solution

is  $f = -e^t/3 + C_1e^{2t} + C_2e^{-2t}$

EXAMPLE 5:  $f'' - 4f = e^{2t}$

The homogeneous solution  $C_1e^{2t} + C_2e^{-2t}$  is the same as before. To get a special solution, we can not use  $Ae^{2t}$  because it is in the kernel of  $D^2 - 4$ . We try  $Ate^{2t}$ , compare coefficients and

get  $f = te^{2t}/4 + C_1e^{2t} + C_2e^{-2t}$

EXAMPLE 6:  $f'' + 4f = e^t$

The homogeneous equation is a harmonic oscillator with solution  $C_1 \cos(2t) + C_2 \sin(2t)$ . To get a special solution, we try  $Ae^t$  compare coefficients and get

$$f = e^t/5 + C_1 \cos(2t) + C_2 \sin(2t)$$

EXAMPLE 7:  $f'' + 4f = \sin(t)$

The homogeneous solution  $C_1 \cos(2t) + C_2 \sin(2t)$  is the same as in the last example. To get a special solution, we try  $A \sin(t) + B \cos(t)$  compare coefficients (because we have only even derivatives, we can even try  $A \sin(t)$ ) and get

$$f = \sin(t)/3 + C_1 \cos(2t) + C_2 \sin(2t)$$

EXAMPLE 8:  $f'' + 4f = \sin(2t)$

The solution  $C_1 \cos(2t) + C_2 \sin(2t)$  is the same as in the last example. To get a special solution, we can not try  $A \sin(t)$  because it is in the kernel of the operator. We try  $At \sin(2t) + Bt \cos(2t)$  instead and compare coefficients

$$f = \sin(2t)/16 - t \cos(2t)/4 + C_1 \cos(2t) + C_2 \sin(2t)$$

EXAMPLE 9:  $f'' + 8f' + 16f = \sin(5t)$

The homogeneous solution is  $C_1 e^{-4t} + C_2 t e^{-4t}$ . To get a special solution, we try  $A \sin(5t) + B \cos(5t)$  compare coefficients and get

$$f = -40 \cos(5t)/41^2 + -9 \sin(5t)/41^2 + C_1 e^{-4t} + C_2 t e^{-4t}$$

EXAMPLE 10:  $f'' + 8f' + 16f = e^{-4t}$

The homogeneous solution is still  $C_1 e^{-4t} + C_2 t e^{-4t}$ . To get a special solution, we can not try  $e^{-4t}$  nor  $t e^{-4t}$  because both are in the kernel. Add an other  $t$  and try with  $At^2 e^{-4t}$ .

$$f = t^2 e^{-4t}/2 + C_1 e^{-4t} + C_2 t e^{-4t}$$

EXAMPLE 11:  $f'' + f' + f = e^{-t}$

By factoring  $D^2 + D + 1 = (D - (1 + \sqrt{3}i)/2)(D - (1 - \sqrt{3}i)/2)$  we get the homogeneous solution  $C_1 e^{-t/2} \cos(\sqrt{3}t/2) + C_2 e^{-t/2} \sin(\sqrt{3}t/2)$ . For a special solution, try  $Ae^{-t}$ . Comparing coefficients gives  $A = 1/13$ .

$$f = e^{-t}/13 + C_1 e^{-t/2} \cos(\sqrt{3}t/2) + C_2 e^{-t/2} \sin(\sqrt{3}t/2)$$