

7.5.2 If  $z = r(\cos \theta + i \sin \theta)$  then  $z^4 = r^4(\cos 4\theta + i \sin 4\theta)$ .

$z^4 = 1$  if  $r = 1$ ,  $\cos 4\theta = 1$  and  $\sin 4\theta = 0$  so  $4\theta = 2k\pi$  for an integer  $k$ , and  $\theta = \frac{k\pi}{2}$ ,

i.e.  $z = \cos\left(\frac{k\pi}{2}\right) + i \sin\left(\frac{k\pi}{2}\right)$ ,  $k = 0, 1, 2, 3$ . Thus  $z = 1, i, -1, -i$ . See Figure 7.25.

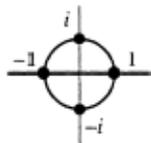


Figure 7.129: for Problem 7.5.2.

**7.6.16N** The eigenvalues are  $4 \pm i$ . We have  $a = 4$ ,  $b = 1$ . The matrix is conjugated to  $S^{-1}AS = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$  with  $S = \begin{bmatrix} 2-i & 2 \\ 1+i & 2 \end{bmatrix}$ , which contains the eigenvectors of  $A$  as columns.

7.5.26  $f_A(\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - 2\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda - 2)\lambda = 0$ , so  $\lambda_{1,2} = 1 \pm i$ ,  $\lambda_3 = 2$ ,  $\lambda_4 = 0$ .

7.5.32 a  $\vec{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} 0.6a(t) + 0.1m(t) + 0.5s(t) \\ 0.2a(t) + 0.7m(t) + 0.1s(t) \\ 0.2a(t) + 0.2m(t) + 0.4s(t) \end{bmatrix}$  so  $A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$ .

Note that  $A$  is a regular transition matrix.

b By Exercise 30,  $\lim_{t \rightarrow \infty} (A^t) = [\vec{v} \vec{v} \vec{v}]$ , where  $\vec{v}$  is the unique eigenvector of  $A$  with eigenvalue

1 and column sum 1. We find that  $\vec{v} = \begin{bmatrix} 0.4 \\ 0.35 \\ 0.25 \end{bmatrix}$ .

Now  $\lim_{t \rightarrow \infty} \vec{x}(t) = \lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \left( \lim_{t \rightarrow \infty} A^t \right) \vec{x}_0 = [\vec{v} \vec{v} \vec{v}] \vec{x}_0 = \vec{v}$ , since the components of  $\vec{x}_0$  add up to 1. The market shares approach 40%, 35%, and 25%, respectively, regardless of the initial shares.

$$7.5.38 \text{ a } C_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C_4^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, C_4^4 = I_4, \text{ then } C_4^{4+k} = C_4^k.$$

Figure 7.31 illustrates how  $C_4$  acts on the basis vectors  $e_i$ .

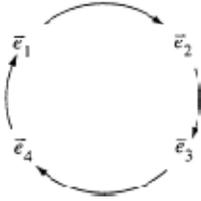


Figure 7.135: for Problem 7.5.38a.

b The eigenvalues are  $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i,$  and  $\lambda_4 = -i,$  and for each eigenvalue  $\lambda_k, \vec{v}_k = \begin{bmatrix} \lambda_k^3 \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix}$  is an associated eigenvector.

$$c \quad M = aI_4 + bC_4 + cC_4^2 + dC_4^3$$

If  $\vec{v}$  is an eigenvector of  $C_4$  with eigenvalue  $\lambda,$  then  $M\vec{v} = a\vec{v} + b\lambda\vec{v} + c\lambda^2\vec{v} + d\lambda^3\vec{v} = (a + b\lambda + c\lambda^2 + d\lambda^3)\vec{v},$  so that  $\vec{v}$  is an eigenvector of  $M$  as well, with eigenvalue  $a + b\lambda + c\lambda^2 + d\lambda^3.$

The eigenbasis for  $C_4$  we found in part b is an eigenbasis for all circulant  $4 \times 4$  matrices.

7.5.50 The eigenvalues are 0, 0, 1. Since the kernel is always two-dimensional, with basis  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$  the matrix is diagonalizable for all values of constant  $a.$

**7.5.36 a** The entries in the first row are age-specific birth rates and the entries just below the diagonal are age-specific survival rates. For example, the entry 1.6 in the first row tells us that during the next 15 years the people who are 15–30 years old today will *on average* have 1.6 children (3.2 per couple) who will survive to the next census. The entry 0.53 tells us that 53% of those in the age group 45–60 today will still be alive in 15 years (they will then be in the age group 60–75).

**b** Using technology, we find the largest eigenvalue  $\lambda_1 = 1.908$  with associated eigenvector

$$\vec{v}_1 \approx \begin{bmatrix} 0.574 \\ 0.247 \\ 0.115 \\ 0.047 \\ 0.014 \\ 0.002 \end{bmatrix}.$$

The components of  $\vec{v}_1$  give the distribution of the population among the age groups in the long run, assuming that current trends continue.  $\lambda_1$  gives the factor by which the population will grow in the long run in a period of 15 years; this translates to an annual growth factor of  $\sqrt[15]{1.908} \approx 1.044$ , or an annual growth of about 4.4%.

**Ch 7.TF.11** T, by Example 6 of Section 7.5

**Ch 7.TF.41** F; Consider a rotation through  $\pi/2$ .