6.2.8 After four row swaps, we end up with an upper triangular matrix B with all 1's along the diagonal, except for a 2 in the bottom right corner. Now $\det(A) = (-1)^4 \det(B) = 2$, by Algorithm 6.2.5b.

6.2.10 $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} \begin{matrix} \\ -I \\ -I \rightarrow \\ -I \\ -I \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{bmatrix} \begin{matrix} \\ \\ -2II \rightarrow \\ -3II \\ -4II \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{bmatrix} \begin{matrix} \\ \\ \rightarrow \\ -3III \\ -6III \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ -4IV \end{matrix} \rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now $\det(A) = \det(B) = 1$ by Algorithm 6.2.5b.

6.2.38 $$\det(A^TA) \underset{\substack{\uparrow \\ \text{Theorem 6.2.6}}}{=} \det(A^T)\det(A) \underset{\substack{\uparrow \\ \text{Theorem 6.2.1}}}{=} [\det(A)]^2 = 9$$

6.2.66N a Using Laplace expansion with respect to the first row we get $d_n = d_{n-1} - d_{n-2}$. Proof $d_n = \det(M_n) = \det(M_{n-1}) - \det \begin{bmatrix} 1 & * \\ 0 & M_{n-2} \end{bmatrix} = \det(M_{n-1}) - \det(M_{n-2}) =$

$d_{n-1} - d_{n-2}$.

b $d_1 = 1, d_2 = 0, d_3 = -1, d_4 = -1, d_5 = 0, d_6 = 1, d_7 = 1, d_8 = 0.$

c $d_4 = -d_1$ and $d_5 = -d_2$, the formula $d_{n+3} = -d_n$ holds for all positive integers n. (One can give a formal prove by induction). Now $d_{n+6} = -d_{n+3} = -(-d_n) = d_n$ for all positive integers n, meaning that the sequence d_n has a period of six.

d $d_{100} = d_4 = -d_1 = -1.$

6.3.12 Denote the columns by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$. From Theorem 6.3.4 and Exercise 6.3.8 we know that $|\det(A)| \leq \|\vec{v}_1\|\|\vec{v}_2\|\ \|\vec{v}_3\|\|\vec{v}_4\|$; equality holds if the columns are orthogonal.

Since the entries of the $\vec{v}_i$ are 0, 1, and -1, we have $\|\vec{v}_i\| \leq \sqrt{1+1+1+1} = 2$. Therefore, $|\det A| \leq 16$.

To build an example where $\det(A) = 16$ we want all 1's and -1's as entries, and the columns need to be orthogonal. A little experimentation produces $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ (there are other solutions). Note that we need to *check* that $\det(A) = 16$ (and not -16).

6.2.50 There are many ways to do this problem; here is one possible approach:

Subtracting the second to last row from the last, we can make the last row into

$[0 \quad 0 \quad \cdots \quad 0 \quad 1]$.

Now expanding along the last row we see that $\det(M_n) = \det(M_{n-1})$.

Since $\det(M_1) = 1$ we can conclude that $\det(M_n) = 1$ for all n.

6.2.44 a We claim that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq \vec{0}$ if and only if the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent. If the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, then we can find a basis $\vec{x}, \vec{v}_2, \ldots, \vec{v}_n$ of $\mathbb{R}^n$ (any vector $\vec{x}$ that is not in span $(\vec{v}_2, \ldots, \vec{v}_n)$ will do). Then $\vec{x} \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{x}\,\vec{v}_2 \cdots \vec{v}_n] \neq 0$, so that $\vec{v}_2 \times \cdots \times \vec{v}_n \neq \vec{0}$. Conversely, suppose that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq 0$; say the ith component of this vector is nonzero. Then $0 \neq \vec{e}_i \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{e}_i\,\vec{v}_2 \cdots \vec{v}_n]$, so that the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent (being columns of an invertible matrix).

b ith component of $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \det \begin{bmatrix} 1 & 1 & & 1 \\ \vec{e}_i & \vec{e}_2 & \cdots & \vec{e}_n \\ 1 & 1 & & 1 \end{bmatrix} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$

so $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \vec{e}_1$.

c $\vec{v}_i \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{v}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix} = 0$

for any $2 \leq i \leq n$ since the above matrix has two identical columns.

d Compare the ith components of the two vectors:

$\det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix}$ and $\det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_3 & \vec{v}_2 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix}$.

The two determinants differ by a factor of -1 by Theorem 6.2.3b, so that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n = -\vec{v}_3 \times \vec{v}_2 \times \cdots \times \vec{v}_n$.

e $\det[\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \ \vec{v}_2 \ \vec{v}_3 \cdots \vec{v}_n] = (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = |\,\vec{v}_2 \times \cdots \times \vec{v}_n\|^2$

f In Definition 6.1.1 we saw that the "old" cross product satisfies the defining equation of the "new" cross product: $\vec{x} \cdot (\vec{v}_2 \times \vec{v}_3) = \det[\,\vec{x} \ \vec{v}_2 \ \vec{v}_3\,]$.

Ch **6.TF.32** F; Note that $\det(S^T AS) = (\det S)^2 (\det A)$ and $\det(-A) = -(\det A)$ have opposite signs.

Ch **6.TF.34** F; Let $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, for example.