5.1.8 Since $\vec{u} \cdot \vec{v} = 4 - 24 + 20 = 0$, the two vectors enclose a right angle.

5.1.16 You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors $\vec{x}$ that are orthogonal to $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, then we identify the unit vectors among them.

Finding the vectors $\vec{x}$ with $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \vec{x} \cdot \vec{v}_3 = 0$ amounts to solving the system

$$\begin{bmatrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{bmatrix}$$

(we can omit all the coefficients $\frac{1}{2}$).

The solutions are of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$.

Since $\|\vec{x}\| = 2|t|$, we have a unit vector if $t = \frac{1}{2}$ or $t = -\frac{1}{2}$. Thus there are two possible choices for $\vec{v}_4$:

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

5.1.18 a $\|\vec{x}\|^2 = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$ (use the formula for a geometric series, with $a = \frac{1}{4}$), so that $\|\vec{x}\| = \frac{2}{\sqrt{3}} \approx 1.155$.

b If we let $\vec{u} = (1, 0, 0, \ldots)$ and $\vec{v} = \left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$, then

$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \arccos \frac{1}{\frac{2}{\sqrt{3}}} = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6} (= 30^\circ)$.

c $\vec{x} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots, \frac{1}{\sqrt{n}}, \cdots\right)$ does the job, since the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges (a fact discussed in introductory calculus classes).

d If we let $\vec{v} = (1, 0, 0, \ldots), \vec{x} = \left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$ and $\vec{u} = \frac{\vec{x}}{\|\vec{x}\|} = \frac{\sqrt{3}}{2}\left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$ then

$\text{proj}_L \vec{v} = (\vec{u} \cdot \vec{v})\vec{u} = \frac{3}{4}\left(1, \frac{1}{2}, \frac{1}{4}, \cdots\right)$.

5.1.20 On the line L spanned by $\vec{x}$ we want to find the vector $m\vec{x}$ closest to $\vec{y}$ (that is, we want $\|m\vec{x}-\vec{y}\|$ to be minimal). We want $m\vec{x}-\vec{y}$ to be perpendicular to L (that is, to $\vec{x}$), which means that $\vec{x}\cdot(m\vec{x}-\vec{y})=0$ or $m(\vec{x}\cdot\vec{x})-\vec{x}\cdot\vec{y}=0$ or $m=\frac{\vec{x}\cdot\vec{y}}{\vec{x}\cdot\vec{x}}\approx\frac{4182.9}{198.53^2}\approx 0.106$.

Recall that the correlation coefficient r is $r=\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}$, so that $m=\frac{\|\vec{y}\|}{\|\vec{x}\|}r$. See Figure 5.3.

5.1.28 Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1=\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\vec{u}_2=\frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix},\vec{u}_3=\frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}.$$

Now we can use Theorem 5.1.5, with $\vec{x}=\vec{e}_1$: $\text{proj}_V\vec{x}=(\vec{u}_1\cdot\vec{x})\vec{u}_1+(\vec{u}_2\cdot\vec{x})\vec{u}_2+(\vec{u}_3\cdot\vec{x})\vec{u}_3=$
$\frac{1}{4}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix}$.

5.1.38 Since $\vec{v}_1$ and $\vec{v}_2$ are unit vectors, the condition $\vec{v}_1\cdot\vec{v}_2=\|\vec{v}_1\|\|\vec{v}_2\|\cos(\alpha)=\cos(\alpha)=\frac{1}{2}$ implies that $\vec{v}_1$ and $\vec{v}_2$ enclose an angle of 60° $(=\frac{\pi}{3})$. The vectors $\vec{v}_1$ and $\vec{v}_3$ enclose an angle of 60° as well.
In the case $n=2$ there are two possible scenarios: either $\vec{v}_2=\vec{v}_3$, or $\vec{v}_2$ and $\vec{v}_3$ enclose an angle of 120°. Therefore, either $\vec{v}_2\cdot\vec{v}_3=1$ or $\vec{v}_2\cdot\vec{v}_3=\cos(120^\circ)=-\frac{1}{2}$.
In the case $n=3$, the vectors $\vec{v}_2$ and $\vec{v}_3$ could enclose any angle between 0° (if $\vec{v}_2=\vec{v}_3$) and 120°, as illustrated in Figure 5.7. We have $-\frac{1}{2}\le\vec{v}_2\cdot\vec{v}_3\le 1$.

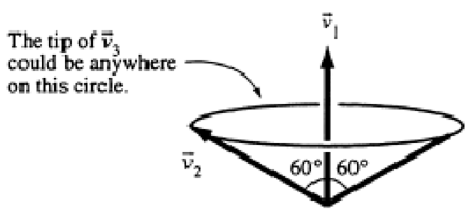

Figure 5.79: for Problem 5.1.38.

For example, consider $\vec{v}_1=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}0\\\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{bmatrix}$, $\vec{v}_3=\begin{bmatrix}\left(\frac{\sqrt{3}}{2}\right)\cos\theta\\\left(\frac{\sqrt{3}}{2}\right)\sin\theta\\\frac{1}{2}\end{bmatrix}$

Note that $\vec{v}_2\cdot\vec{v}_3=\left(\frac{3}{4}\right)\sin\theta+\frac{1}{4}$ could be anything between $-\frac{1}{2}$ (when $\sin\theta=-1$) and 1 (when $\sin\theta=1$), as claimed.

If n exceeds three, we can consider the orthogonal projection $\vec{w}$ of $\vec{v}_3$ onto the plane E spanned by $\vec{v}_1$ and $\vec{v}_2$.

Since $\text{proj}_{\vec{v}_1}\vec{w}=(\vec{v}_1\cdot\vec{w})\vec{v}_1=\frac{1}{2}\vec{v}_1$, and since $\|\vec{w}\|\le\|\vec{v}_3\|=1$, (by Theorem 5.1.10), the tip of $\vec{w}$ will be on the line segment in Figure 5.8. Note that the angle ϕ enclosed by the vectors $\vec{v}_2$ and $\vec{w}$ is between 0° and 120°, so that $\cos\phi$ is between $-\frac{1}{2}$ and 1.

Therefore, $\vec{v}_2 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{w} = \|\vec{w}\| \cos\phi$ is between $-\frac{1}{2}$ and 1.

This implies that $\angle(\vec{v}_2, \vec{v}_3)$ is between 0° and 120° as well. To see that all these values are attained, add $(n-3)$ zeros to the three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$ given above.

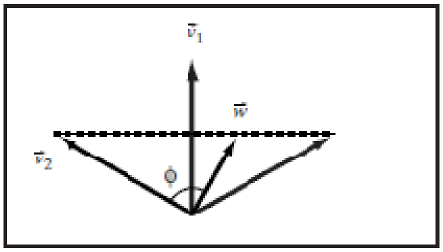

Figure 5.80: for Problem 5.1.38.

5.1.14 The horizontal components of $\vec{F}_1$ and $\vec{F}_2$ are $-\|\vec{F}_1\| \sin\beta$ and $\|\vec{F}_2\| \sin\alpha$, respectively (the horizontal component of $\vec{F}_3$ is zero).

Since the system is at rest, the horizontal components must add up to 0, so that $-\|\vec{F}_1\| \sin\beta + \|\vec{F}_2\| \sin\alpha = 0$ or $\|\vec{F}_1\| \sin\beta = \|\vec{F}_2\| \sin\alpha$ or $\frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} = \frac{\sin\alpha}{\sin\beta}$.

To find $\frac{\overline{EA}}{\overline{EB}}$, note that $\overline{EA} = \overline{ED}\tan\alpha$ and $\overline{EB} = \overline{ED}\tan\beta$ so that $\frac{\overline{EA}}{\overline{EB}} = \frac{\tan\alpha}{\tan\beta} = \frac{\sin\alpha}{\sin\beta} \cdot \frac{\cos\beta}{\cos\alpha} = \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} \frac{\cos\beta}{\cos\alpha}$. Since α and β are two distinct acute angles, it follows that $\frac{\overline{EA}}{\overline{EB}} \neq \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|}$, so that Leonardo was mistaken.