4.1.3 This subset V is a subspace of P_2:

- The neutral element $f(t) = 0$ (for all t) is in V since $f'(1) = f(2) = 0$.
- If f and g are in V (so that $f'(1) = f(2)$ and $g'(1) = g(2)$), then

  $(f+g)'(1) = (f'+g')(1) = f'(1)+g'(1) = f(2)+g(2) = (f+g)(2)$, so that $f+g$ is in V.
- If f is in V (so that $f'(1) = f(2)$) and k is any constant, then $(kf)'(1) = (kf')(1) = kf'(1) = kf(2) = (kf)(2)$, so that kf is in V.

If $f(t) = a + bt + ct^2$ then $f'(t) = b + 2ct$, and f is in V if

$f'(1) = b + 2c = a + 2b + 4c = f(2)$, or $a + b + 2c = 0$. The general element of V is of the form $f(t) = (-b-2c) + bt + ct^2 = b(t-1) + c(t^2-2)$, so that $t-1, t^2-2$ is a basis of V.

4.1.4 This subset V is a subspace of P_2:

- The neutral element $f(t) = 0$ (for all t) is in V since $\int_0^1 0\,dt = 0$.
- If f and g are in V $\left(\text{so that } \int_0^1 f = \int_0^1 g = 0\right)$ then $\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0$, so that $f+g$ is in V.
- If f is in V $\left(\text{so that } \int_0^1 f = 0\right)$ and k is any constant, then $\int_0^1 kf = k\int_0^1 f = 0$, so that kf is in V.

If $f(t) = a + bt + ct^2$ then $\int_0^1 f(t)dt = \left[at + \frac{b}{2}t^2 + \frac{c}{3}t^3\right]_0^1 = a + \frac{b}{2} + \frac{c}{3} = 0$ if $a = -\frac{b}{2} - \frac{c}{3}$.

The general element of V is $f(t) = \left(-\frac{b}{2} - \frac{c}{3}\right) + bt + ct^2 = b\left(t - \frac{1}{2}\right) + c\left(t^2 - \frac{1}{3}\right)$, so that $t - \frac{1}{2}$, $t^2 - \frac{1}{3}$ is a basis of V.

4.1.14 Yes

- $(0, 0, 0, \ldots, 0, \ldots)$ converges to 0.
- If $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0$.
- If $\lim_{n\to\infty} x_n = 0$ and k is any constant, then $\lim_{n\to\infty}(kx_n) = k\lim_{n\to\infty} x_n = 0$.

4.1.38 We look at similar cases mentioned in Exercise 37, and see that the different possibilities occur when all four entries are different ($\dim(V)=4$), when exactly two are the same, but the other two are different ($\dim(V)=6$), when exactly three are the same ($\dim(V)=10$), when all four are the same ($\dim(V)=16$) and when two of the terms of B are equal, and the other two diagonal terms of B are also equal, but different from the first pair ($\dim(V)=8$).

4.1.50 Using Example 18 as a guide, we first look for solutions of the form $f(x)=e^{kx}$. It is required that $f''(x)+8f'(x)-20f(x)=k^2e^{kx}+8ke^{kx}-20e^{kx}=0$ for all x, or $k^2+8k-20=(k-2)(k+10)=0$. Thus $k=2$ or $k=-10$. By Theorem 4.1.5, the solutions of the differential equation are of the form $f(x)=c_1e^{2x}+c_2e^{-10x}$, where c_1 and c_2 are arbitrary constants.

4.1.52 We have to find constants a and b such that the functions e^{-x} and e^{-5x} are solutions of the differential equation $f''(x)+af'(x)+bf(x)=0$. Thus it is required that $e^{-x}-ae^{-x}+be^{-x}=0$, or $1-a+b=0$, and also that $25-5a+b=0$. The solution of this system of two equations in two unknowns is $a=6, b=5$, so that the desired differential equation is $f''(x)+6f'(x)+5f(x)=0$.

4.1.42 Let B be a matrix such that dim(ker(B))= k. Then, it is required that the columns of A contain only vectors in the kernel of B. Thus, each column of A can be written as: $c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_k\vec{v}_k$, where the vectors $\vec{v}_i$ form as basis of the kernel of B. Thus, each of the n columns in A involves k arbitrary constants, and matrix A involves nk arbitrary constants overall. The space of matrices A has dimension nk, where k is an integer in the range $[0,n]$.

4.1.56 Argue indirectly and assume that the space V of infinite sequences is finite-dimensional, with $\dim(V)=n$. According to the solution to Exercise 57, there can be at most n linearly independent elements in V. But here is our contradiction: It is easy to give $n+1$ linearly independent infinite sequences, namely,

$(1,0,0,0,\ldots),(0,1,0,0,\ldots),(0,0,1,0,\ldots),\ldots,(0,0,0,\ldots,0,1,0,\ldots)$; in the last sequence the 1 is in the $(n+1)$th place.