5.5.4 a For column vectors $\vec{v}, \vec{w}$, we have $\langle \vec{v}, \vec{w} \rangle = \text{trace}(\vec{v}^T \vec{w}) = \text{trace}(\vec{v} \cdot \vec{w}) = \vec{v} \cdot \vec{w}$, the dot product.

b For row vectors $\vec{v}, \vec{w}$, the $ij$th entry of $\vec{v}^T \vec{w}$ is $v_i w_j$, so that $\langle \vec{v}, \vec{w} \rangle = \text{trace}(\vec{v}^T \vec{w}) = \sum_{i=1}^{m} v_i w_i = \vec{v} \cdot \vec{w}$, again the dot product.

5.5.10 A function $g(t) = a + bt + ct^2$ is orthogonal to $f(t) = t$ if

$$\langle f, g \rangle = \int_{-1}^{1} (at + bt^2 + ct^3)\, dt = [\, \tfrac{a}{2}t^2 + \tfrac{b}{3}t^3 + \tfrac{c}{4}t^4 \,]_{-1}^{1} = \frac{2}{3}b = 0, \text{ that is, if } b = 0.$$

Thus, the functions 1 and $t^2$ form a basis of the space of all functions in $P_2$ orthogonal to $f(t) = t$. To find an *orthonormal* basis $g_1(t), g_2(t)$, we apply Gram-Schmidt. Now $\|1\| = \frac{1}{2}\int_{-1}^{1} 1\, dt = 1$, so that we can let $g_1(t) = 1$. Then $g_2(t) = \frac{t^2 - \langle 1, t^2 \rangle 1}{\|t^2 - \langle 1, t^2 \rangle 1\|} = \frac{t^2 - \frac{1}{3}}{\|t^2 - \frac{1}{3}\|} = \frac{\sqrt{5}}{2}(3t^2 - 1)$

*Answer*: $g_1(t) = 1, g_2(t) = \frac{\sqrt{5}}{2}(3t^2 - 1)$

5.5.16 a We start with the standard basis 1, $t$ and use the Gram-Schmidt process to construct an *orthonormal* basis $g_1(t), g_2(t)$.

$\|1\| = \sqrt{\int_0^1 dt} = 1$, so that we can let $g_1(t) = 1$. Then $g_2(t) = \frac{t-\langle 1,t\rangle 1}{\|t-\langle 1,t\rangle 1\|} = \frac{t-\frac{1}{2}}{\|t-\frac{1}{2}\|} = \sqrt{3}(2t-1)$.

*Summary*: $g_1(t) = 1$ and $g_2(t) = \sqrt{3}(2t-1)$ is an orthonormal basis.

**b** We are looking for $\text{proj}_{P_1}(t^2) = \langle g_1(t), t^2\rangle g_1(t) + \langle g_2(t), t^2\rangle g_2(t)$, by Theorem 5.5.3.

We find that $\langle g_1(t), t^2\rangle = \int_0^1 t^2\,dt = \frac{1}{3}$ and $\langle g_2(t), t^2\rangle = \sqrt{3}\int_0^1 (2t^3 - t^2)\,dt = \frac{\sqrt{3}}{6}$, so that $\text{proj}_{P_1} t^2 = \frac{1}{3} + \frac{1}{2}(2t-1) = t - \frac{1}{6}$. See Figure 5.21.

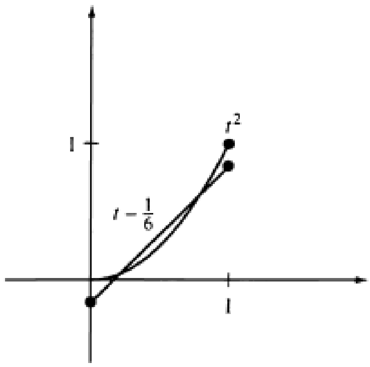



Figure 5.93: for Problem 5.5.16b.

**5.5.22** Apply the Cauchy-Schwarz inequality to $f(t)$ and $g(t) = 1$; note that $\|g\| = 1$:

$|\langle f, g\rangle| \le \|f\|\|g\| = \|f\|$ or $\langle f, g\rangle^2 \le \|f\|^2$ or $\left(\int_0^1 f(t)\,dt\right)^2 \le \int_0^1 (f(t))^2\,dt$.

**5.5.34N a** Note that the graph of $\sqrt{1=t^2}$ is the upper half of the unit circle centered at the origin. Therefore $\int_{-1}^{1} w(t)\ dt = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\ dt = 2\pi$ times the area of the unit circle $= 1$. It can also be shown directly using the substitution $t = \sin(u), dt = \cos(u)\ du$:

$$\int_{-\pi/2}^{\pi/2} \frac{2}{\pi}\sqrt{1-\sin^2(u)}\ \cos(u)\ du = \int_{-\pi/2}^{\pi/2} \frac{2}{\pi}\cos^2(u)\ du = \int_{-\pi/2}^{\pi/2} \frac{2}{\pi}(1+\cos(2u))/2\ du = 1\ .$$

**b** If $f(t) = 1$, we can use a) to see $||f||^2 = \int_{-1}^{1}\ 1^2\frac{2}{\pi}\sqrt{1-\sin^2(u)dt})^{1/2} = 1$. Therefore, by Exercise 5.5.33b, $||f|| = 1$.

**c** $\langle t^2, t^3\rangle = \int_{-1}^{1} t^5 \frac{2}{\pi}\sqrt{1-t^2}dt) = 0$ because we integrate an odd function over $[-1, 1]$. The integral from $[-1, 0]$ and the integral from $[0, 1]$ cancel.

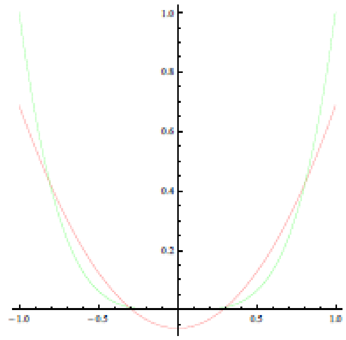
**d** It follows from the definition of this inner product that $\langle t, t\rangle = \langle 1, t^2\rangle = \int_{-1}^{1}\frac{2}{\pi}\sqrt{1-t^2}t^2\ dt = \int_{-\pi/2}^{\pi/2}\sin^2(t)\cos^2(t)\ dt = \int_{-\pi/2}^{\pi/2}(1/4)\sin(2t)^2\ dt = \int_{-\pi/2}^{\pi/2}(1/4)(1-\cos(2t))/2\ dt = (2/pi)\pi/8 = 1/4$. Similarly, we get $\langle t^2, t^2\rangle = \langle t, t^3\rangle = 1/8$ so that $||t|| = \sqrt{1/4} = 1/2$ and $||t^2||^2 = \sqrt{1/8} = \sqrt{2}/4$.

**e** $g_0(t) = 1$. We obtain $g_1(t)$ by normalizing $f_1(t) = t - \langle t, 1\rangle t = t$. Because $||t|| = 1/2$ we have $g_1(t) = 2t$. We get $g_2(t)$ by normalizing $f_2(t) = t^2 - 1\langle t^2, 1\rangle - 2t\langle t^2, 2t\rangle = t^2 - 1/4$ which has norm $||f_2|| = 1/4$. Therefore $g_2(t) = 4t^2 - 1$. Finally, we get $g_3(t)$ by normalizing

$$f_3(t) = t^3 - 1\langle t^3, 1\rangle - 2t\langle t^3, 2t\rangle - (4t^2-1)\langle t^3, 4t^2-1\rangle$$

which simplifies to $t^3 - 3t/5$. It has norm $||f_3|| = \sqrt{213}/8$ so that $g_3(t) = 8t^3 - 4t$. $g_0(t) = 1, g_1(t) = 4t$, $g_2(t) = 4t^2 - 1$, and $g_3(t) = 8t^3 - 4t$.

**f** The polynomial which best approximates $f(t) = t^4$ is $\langle f, g_0\rangle g_0 + \langle f, g_1\rangle g_1 + \langle f, g_2\rangle g_2 + \langle f, g_3\rangle g_3 = -1/16 + (3 * t^2)/4$.




**5.5.20** a $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = [x_1 x_2] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1 + 2x_2 = 0$ when $x_1 = -2x_2$. This is the line spanned by vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

b Since vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are orthogonal, we merely have to multiply each of them with the reciprocal of its norm. Now $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 = [1 \quad 0] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$, so that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a unit vector, and $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\|^2 = [-2 \quad 1] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 4$, so that $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| = 2$. Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$ is an orthonormal basis.