1. The verification that $\cos(nx), \sin(nx), 1/\sqrt{2}$ form an orthonormal family is a straightforward computation, when using the identities provided. For example, $\langle \cos(nx), \sin(mx) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos((n-m)x) - \cos((n+m)x)\,dx$ which is equal to 1 if $n = m$ and equal to 0 if $n \neq m$. The computations can be abbreviated by noting that integrating an odd 2π-periodic function over $[-\pi, \pi]$ is zero because the integral on $[0, \pi]$ cancels with the integral on $[-\pi, 0]$.

2. To get the Fourier series of the function $f(x) = |x|$, note first that this is an **even function** so that it has a cos series. We compute

$$a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{2}{\pi}\int_0^{\pi} x\frac{1}{\sqrt{2}}\,dx = \frac{\pi\sqrt{2}}{2} \, .$$

$$a_n = \langle f, \cos(nx) \rangle = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}[\frac{\cos(n\pi)-1}{n^2}] \, .$$

The Fourier coefficients of $f(x) = 5 + |3x|$ are $a_0 = \frac{3\pi\sqrt{2}}{2} + 5\sqrt{2}$. and $a_n = \frac{6}{\pi}[\frac{\cos(n\pi)-1}{n^2}]$.

3. The Fourier series of $4\cos^2(3x) + 5\sin^2(11x) + 90$ is with

$$\cos^2(2x) = \frac{1+\cos(2x)}{2}$$

$$\sin^2(2x) = \frac{1-\cos(2x)}{2}$$

given as $\boxed{4/2 + 4\cos(6x)/2 - 5\cos(22x)/2 + 5/2 + 90}$. All Fourier coefficients are zero except

$\boxed{a_0 = (189/2)\cdot\sqrt{2} \text{ and } a_6 = 2 \text{ and } a_{22} = -5/2}$.

4. To find the Fourier series of the function $f(x) = |\sin(x)|$, we first note that this is an **even function** so that it has a cos-series. If we integrate from 0 to π and multiply the result by 2, we can take the function $\sin(x)$ instead of $|\sin(x)|$ so that

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin(x)/\sqrt{2}\,dx = \frac{2\sqrt{2}}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{4}{\pi}\frac{1}{1-n^2}$$

for even n and $a_n = 0$ for odd n.
To do the integral, use the trigonometric identities $2\sin(x)\cos(nx) = \sin(x+nx) + \sin(x-nx)$. We have

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos(2x)}{2^2-1} + \frac{\cos(4x)}{4^2-1} + \frac{\cos(6x)}{6^2-1} + \dots\right) \, .$$

5. The square of the length of the function $f(x)$ is 1. The Parseval identity shows that

$$1 = a_0^2 + \sum_{n=1}^{\infty} a_n^2 = (\sqrt{2}\frac{2}{\pi})^2 + \frac{16}{\pi^2}[\frac{1}{(2^2-1)^2} + \frac{1}{(4^2-1)^2} + \frac{1}{(6^2-1)^2} + \cdots]$$

so that the sum is $\boxed{\pi^2/16 - 1/2}$.

6. To solve the heat equation $f_t = 8f_{xx}$ on $[0, \pi]$ with the initial condition $f(x, 0) = |\sin(3x)|$, we make a Fourier expansion of $|\sin(3x)|$:

$$|\sin(3x)| = \sum_{n=1}^{\infty} b_n \sin(nx)$$

and can immediately write down the solution

$$f(x, t) = \sum_{n=1}^{\infty} b_n e^{-8n^2 t}\sin(nx) \, .$$

Now to the Fourier series: note that $\sin(3x)$ is nonnegative on $[0, \pi/3]$ and $[2\pi/3, \pi]$ so that it agrees there with the function $|\sin(3x)|$. On the interval $[\pi/3, 2\pi/3]$ however the function $\sin(3x)$ is negative or zero so that $-\sin(3x)$ is nonnegative there. We get therefore the Fourier coefficients as

$$b_n = \frac{2}{\pi}[\int_0^{\pi/3} \sin(3x)\sin(nx)\,dx - \int_{\pi/3}^{2\pi/3} \sin(3x)\sin(nx)\,dx + \int_{2\pi/3}^{\pi} \sin(3x)\sin(nx)\,dx] \, .$$

We use the identity

$$2\sin(nx)\sin(my) = \cos(nx - my) - \cos(nx + my)$$

to solve these integrals:

$$b_n = \frac{2}{\pi}[(\frac{-3\sin(\frac{n\pi}{3})}{n^2-9}) - (\frac{3\sin(\frac{n\pi}{3})}{n^2-9} + \frac{3\sin(\frac{2n\pi}{3})}{n^2-9}) + (\frac{-3\sin(\frac{2n\pi}{3})}{n^2-9})]$$

which can be simplied to

$$\boxed{\frac{6}{\pi}[\frac{(1+2\cos(n\pi/3))^2\sin(\frac{n\pi}{3})}{n^2-9}]}.$$

The case $n = 3$ might look problematic at first, but the limit still exists.

7. The operator $D^6 + D^2$ has the eigenvectors $\sin(nx)$ with eigenvalues $-n^6 - n^2$. With initial condition $f(x) = b_n\sin(nx)$ we have the solution $b_n\sin(nx)e^{(-n^6-n^2)t}$. The function $|\sin(x)|$ is continued as an odd function so that we have to compute the Fourier series of $\sin(x)$ which is already the Fourier series. The general solution is $\sum_{n=1}^{\infty} b_n\sin(nx)e^{(-n^6-n^2)t}$, where $b_n = 1$ for $n = 1$ and 0 else. We can write it as $\boxed{e^{-2t}\sin(x)}$. Note that we do not do a cos expansion and continued the function $|\sin(x)|$ as and **odd function**. We could have solved the problem also with a cos expansion. This is not wrong. It is just much more work.

8. Because the initial condition is zero on the interval $[\pi/2, \pi]$, we have to integrate from 0 to $\pi/2$ only. The Fourier coefficients of the function $g(x)$ can be computed using one of the trigonometric identities you find on the first page of the handout:

$$\frac{2}{\pi}\int_0^{\pi/2} \sin(2x)\sin(nx)\,dx = \frac{-4}{\pi(n^2-4)}\sin(n\pi/2) \, .$$

The Fourier series of the initial position $f(x) = 0$ of the string is equal to zero by assumption. The solution of the wave equation is

$$f(x, t) = \sum_{n=1}^{\infty} \frac{-4}{\pi(n^2-4)}\sin(n\pi/2)\sin(nx)\sin(nt)\frac{1}{n} \, .$$

The solution also exists for $n = 2$, where it is $1/2$ (which can best be seen by evaluating the original integral $\frac{2}{\pi}\int_0^{\pi/2} \sin^2(2x)\,dx = 1/2$).

9. The general solution of the homogeneous equation with the function at rest initially is is $u_h(t,x) = \sum_n b_n \sin(nx)\cos(nt) = 11\cos(5t)\sin(5x) + 23\cos(7t)\sin(7x)$. A particular solution which is zero at 0 is $u_p(t,x) = -(1/4)\cos(2t) - (1/25)\cos(5t) + 1/4 + 1/25$. Now fix the Fourier coefficients. We end up with

$$u(t,x) = 11\sin(5x)\cos(5t) + 23\sin(7x)\cos(7t) - (1/4)\cos(2t) - (1/25)\cos(5t) + 1/4 + 1/25$$

which is the solution to the differential equation.

10. a) If the function is $\text{sign}(xy)$, we have

$$b_{nm} = \frac{4}{\pi^2}\int_0^\pi \int_0^\pi \sin(nx)\sin(my)\; dy\; dx$$

which is

$$\frac{4}{\pi^2}\left(-\frac{\cos(nx)}{n}\Big|_0^\pi\right)\left(-\frac{\cos(ny)}{n}\Big|_0^\pi\right) = \frac{16}{\pi^2}\frac{1}{nm} \; .$$

The Fourier coefficients are $\boxed{\frac{16}{nm\pi^2}}$ if n, m are both odd and zero else.

b) Since every initial condition $u = b_{nm}\sin(nx)\sin(my)$ satisfies the ordinary differential equation $u_t = (-n^2 - m^2)u$ with solution $u(t) = e^{-n^2-m^2}u(0) = e^{-n^2-m^2}b_{nm}\sin(nx)\sin(my)$. we can add up a linear combination of such solutions and get

$$u(x,y,t) = \sum_{n,m=1}^{\infty} b_{nm}e^{-(n^2+m^2)t}\sin(nx)\sin(my) \; .$$

With the Fourier coefficients computed in part a), we have the final answer $\boxed{u(x,y,t) = \sum_{n,m=odd}^{\infty} \frac{16e^{-(n^2+m^2)t}}{\pi^2 nm}\sin(nx)\sin(my)}$.