Math21b

Review to second midterm

Spring 2009

Oliver Knill

Background: Fantasia movie and Laser music

Orthonormal basis

$$\{ w_1, w_2, \ldots, w_n \}$$

orthonormal basis

Build matrix A from columns

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} / 2$$

to get an **orthogonal** matrix. It satisfies:

$$A^T A = I$$

Also the following matrix has an orthonormal set of vectors as columns.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} / 2$$

we have

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But the matrix A is not an orthogonal matrix.

Orthogonal transformation

$$T(x) = A\,x$$

where A is an orthogonal matrix.

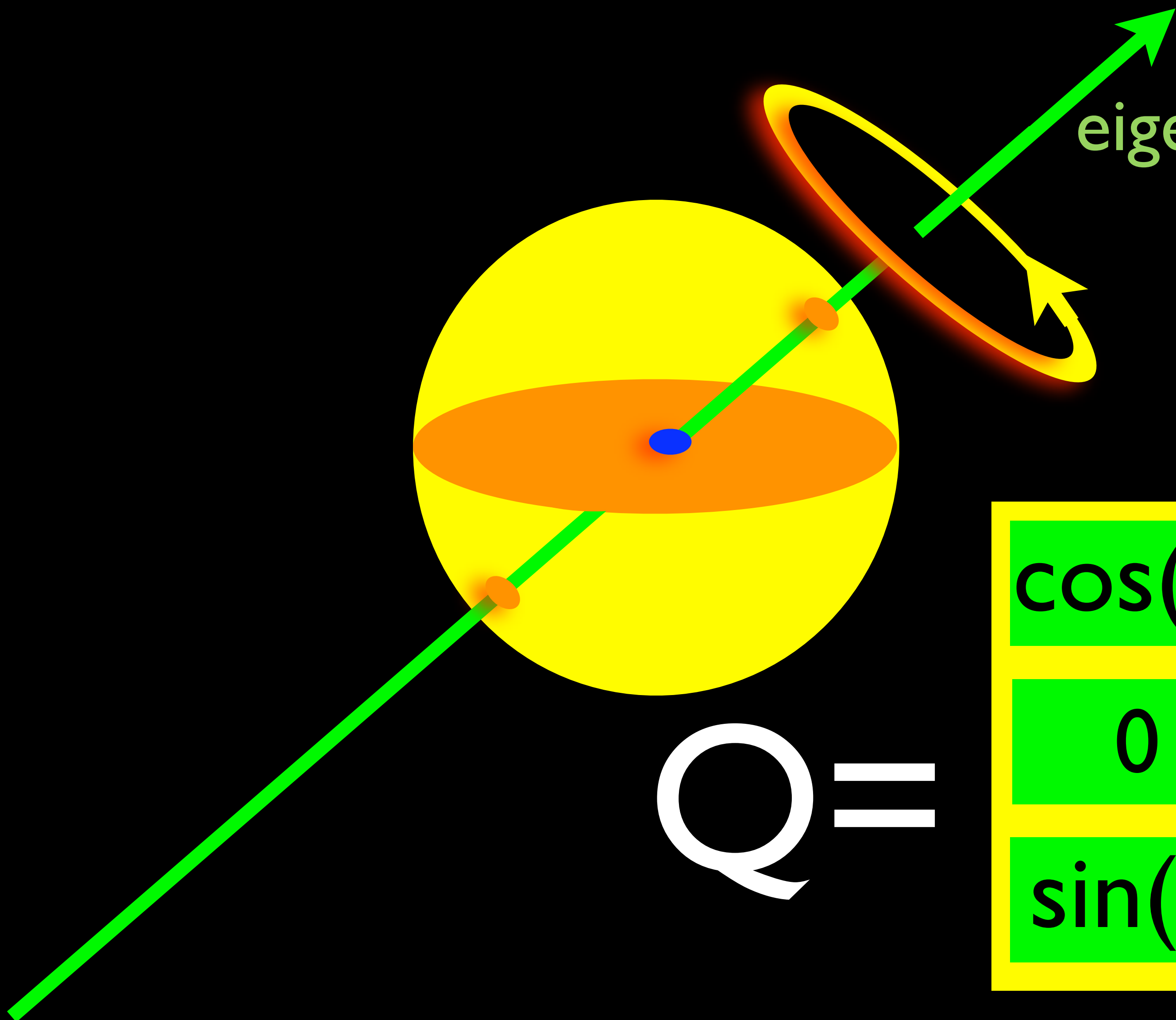
A orthogonal:

$$A^T A = I$$

- typically reflections or rotations
- preserve length and angles
- column vectors of Q form orthonormal basis
- determinant of A is either 1 or -1 (is not equivalent however to orthogonality)

Rotations

$\det(Q) = 1$



example:

$$Q = \begin{pmatrix} \cos(t) & 0 & -\sin(t) \\ 0 & 1 & 0 \\ \sin(t) & 0 & \cos(t) \end{pmatrix}$$

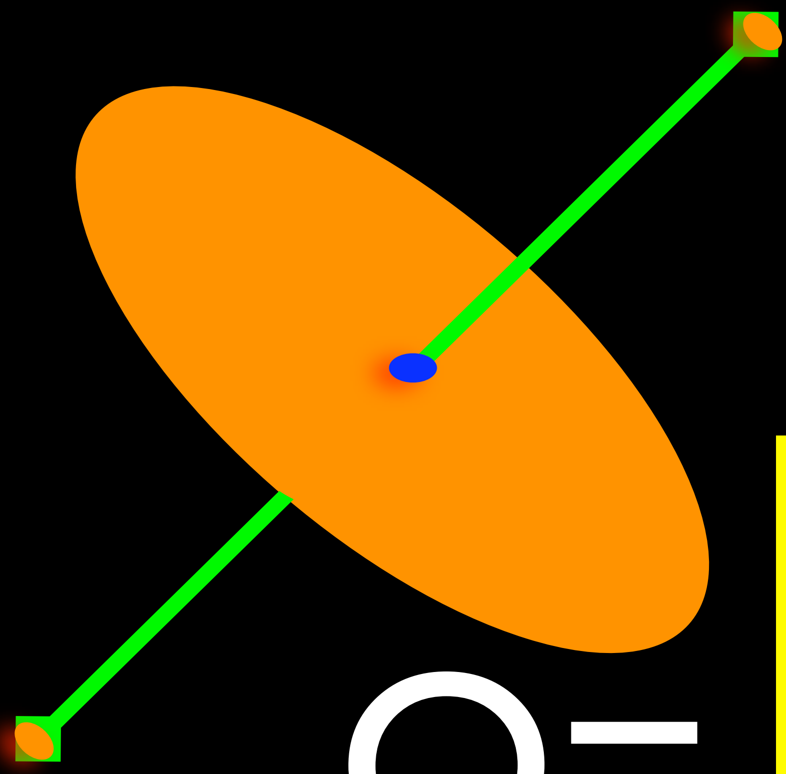
Reflections

det(Q)=-1

if reflected at odd dimensional space

det(Q)=+1

if reflected at even dimensional space



example:

$$Q = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ \sin(t) & -\cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

don't confuse orthogonal transformations
with orthogonal projections
which satisfy

$$A^2 = A$$

and are in general
not invertible

Example 1:

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which of the two matrices is an orthogonal matrix?

1	1	-1	1
1	1	1	-1
1	-1	-1	-1
1	-1	1	1

/2

1	1	-1	-1
1	1	-1	1
1	-1	-1	-1
1	-1	-1	1

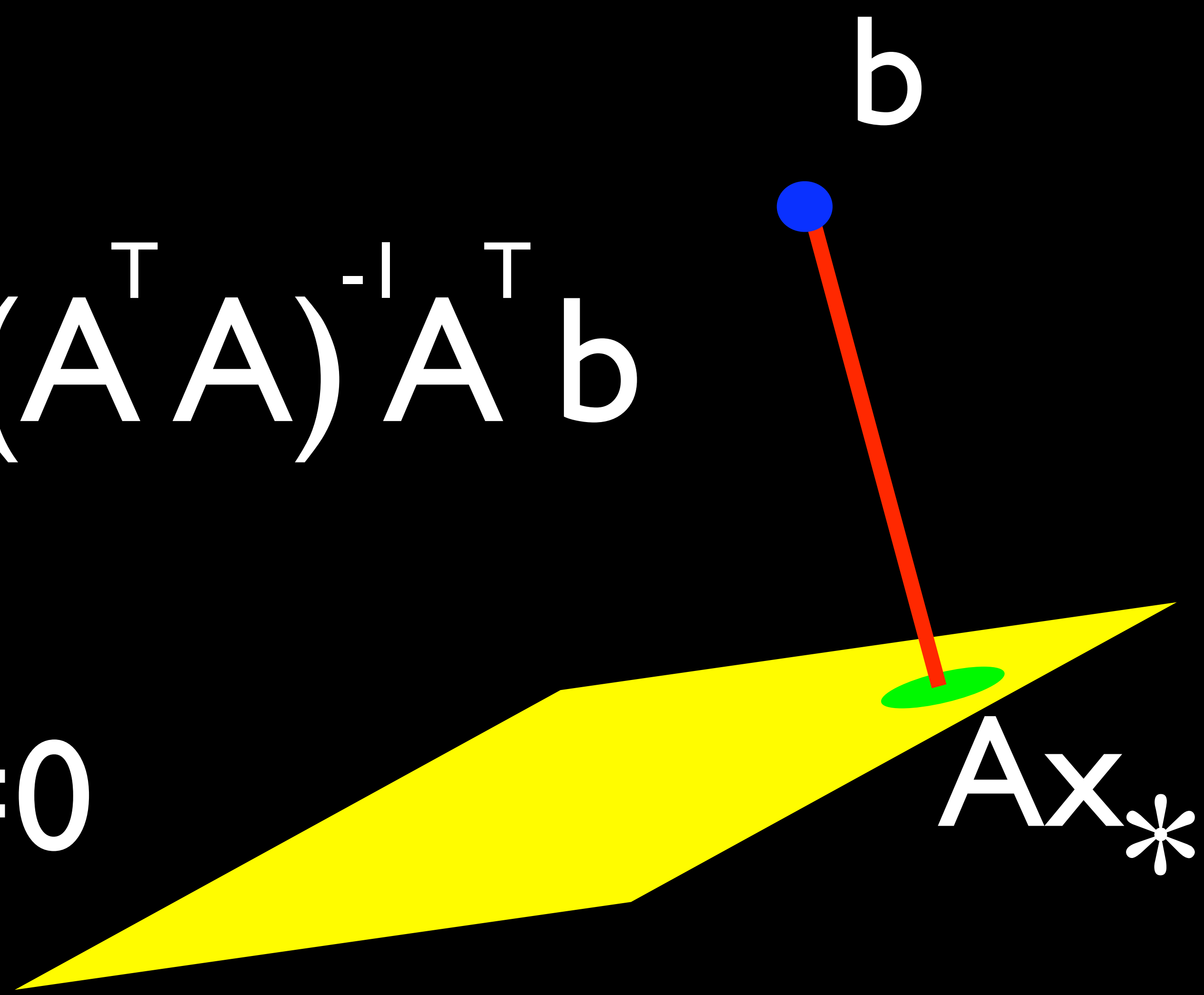
/2

Least Square Solution

solve $Ax=b$ best

$$x_* = (A^T A)^{-1} A^T b$$

$$A^T (Ax-b)=0$$



Projection

$$P\,y = A(A^T A)^{-1} A^T y$$

simplifies to

$$P\,y = A\,A^T y$$

if columns of A are orthonormal.

blackboard problem

Find the matrix of a projection onto the span of the two vectors

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

answer:

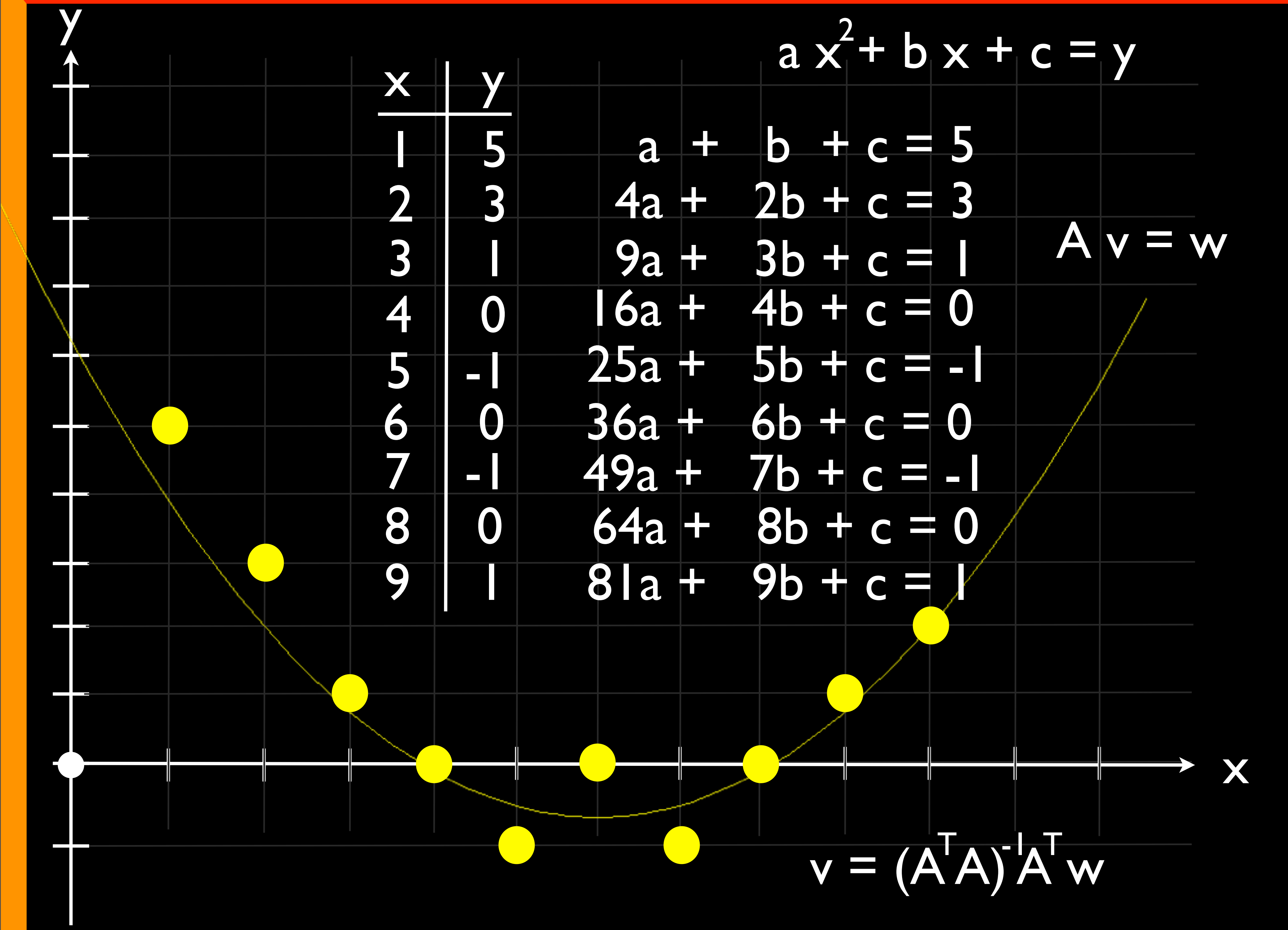
$$P = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ 2 & 0 & 4 & 0 & 2 \\ 0 & 3 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 & 1 \end{bmatrix} / 6$$

Data fitting

To fit data with a set of functions, just plug in the data

$$y_i = f_k(x_i) \;.$$

Here is an example with functions $1, x, x^2$



$$a x^2 + b x + c = y$$

x	y
1	5
2	3
3	1
4	0
5	-1
6	0
7	-1
8	0
9	1

$$\begin{aligned}
a + b + c &= 5\\
4a + 2b + c &= 3\\
9a + 3b + c &= 1\\
16a + 4b + c &= 0\\
25a + 5b + c &= -1\\
36a + 6b + c &= 0\\
49a + 7b + c &= -1\\
64a + 8b + c &= 0\\
81a + 9b + c &= 1
\end{aligned}$$

$$A v = w$$

$$v = (A^T A)^{-1} A^T w$$

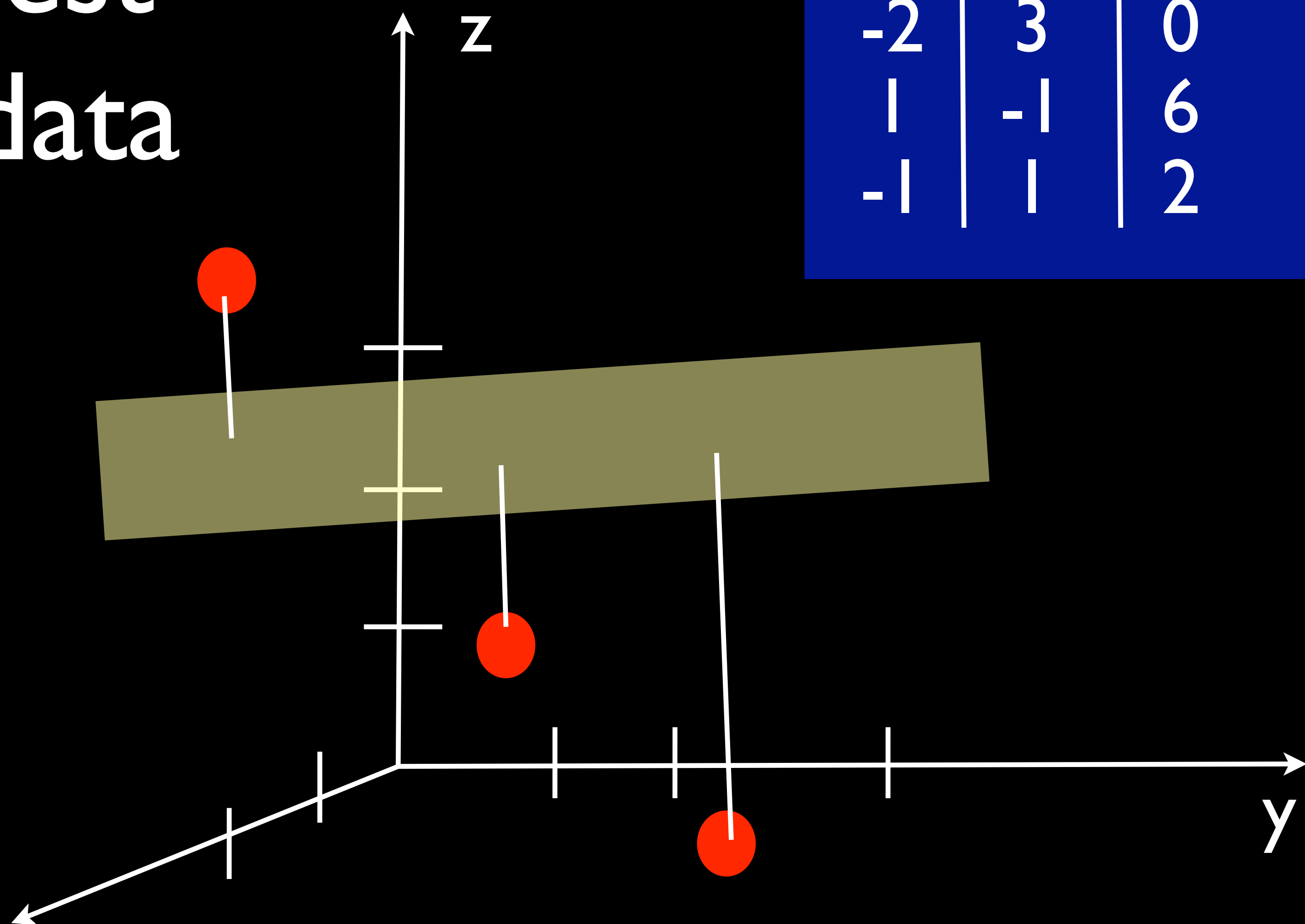

too many
data!
movie: amadeus

blackboard problem

Find the plane which best fits the data

$z=ax+by$

x	y	z
-2	3	0
1	-1	6
-1	1	2



Solution:

x	y	z
-2	3	0
1	-1	6
-1	1	2

$$A^T = \begin{bmatrix} 3 & -1 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

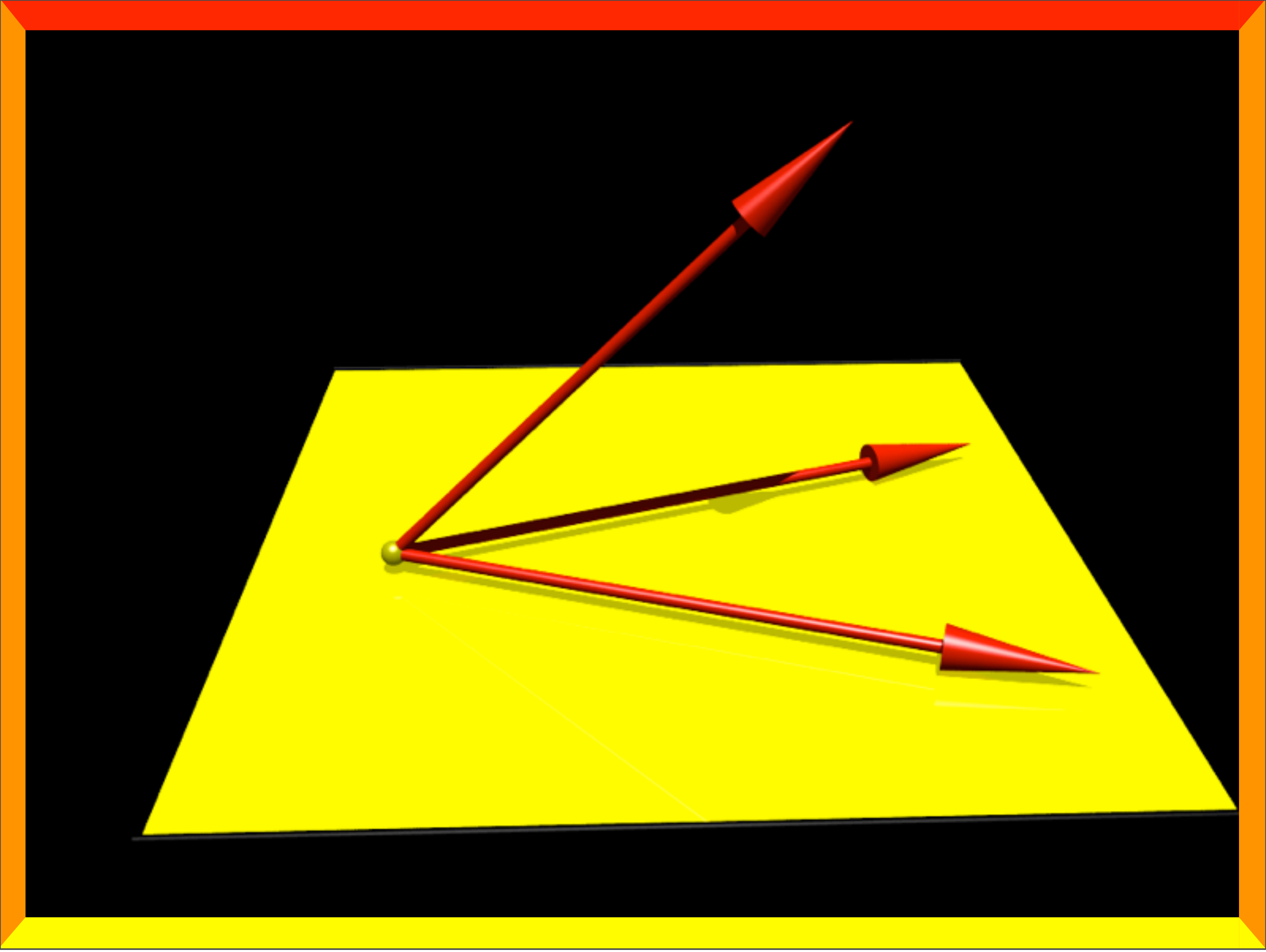
$$A^T b = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

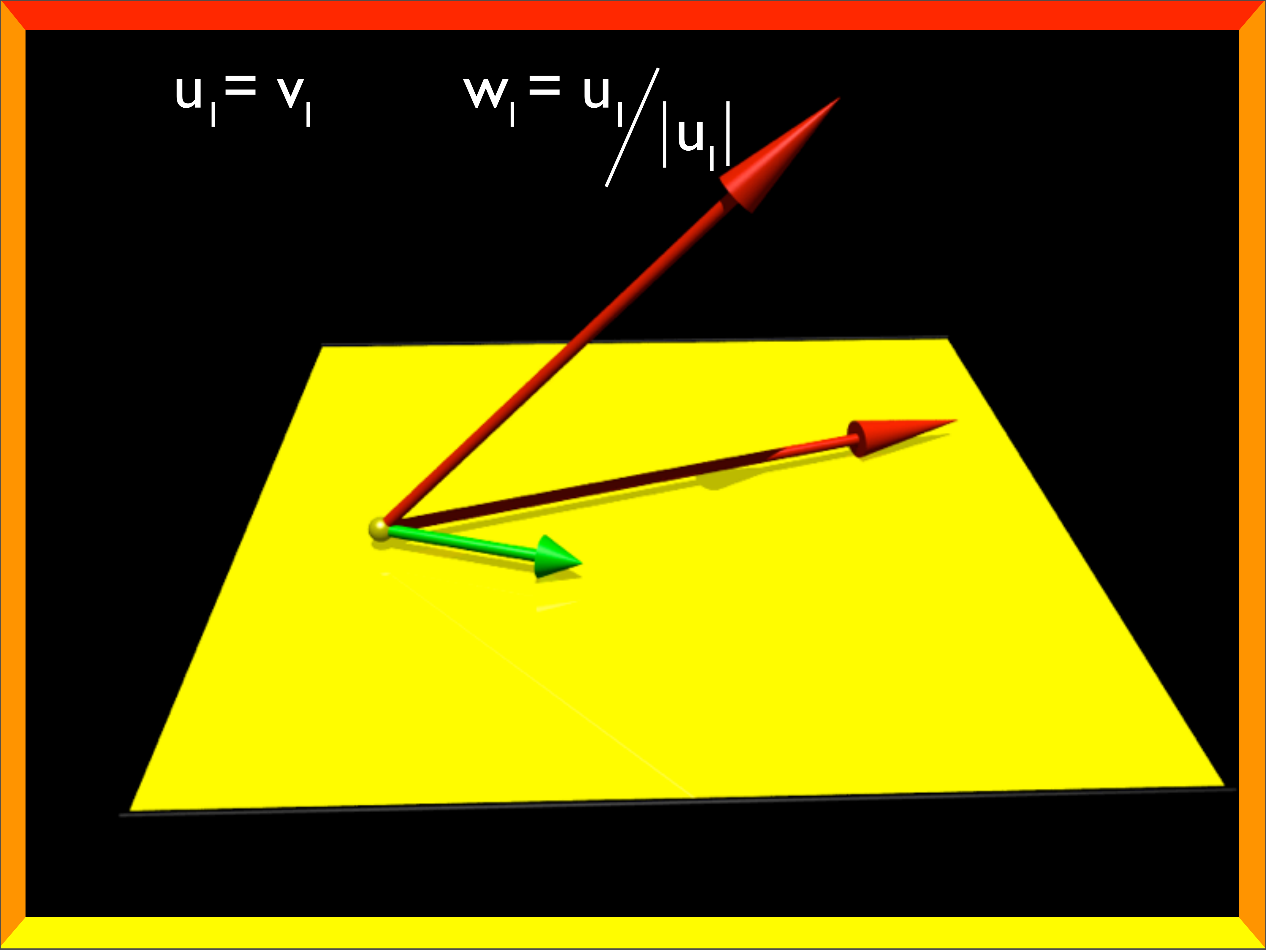
$$(A^T A)^{-1} A^T b = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 6 & -8 \\ -8 & 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 11 & 8 \\ 8 & 6 \end{bmatrix} /2$$

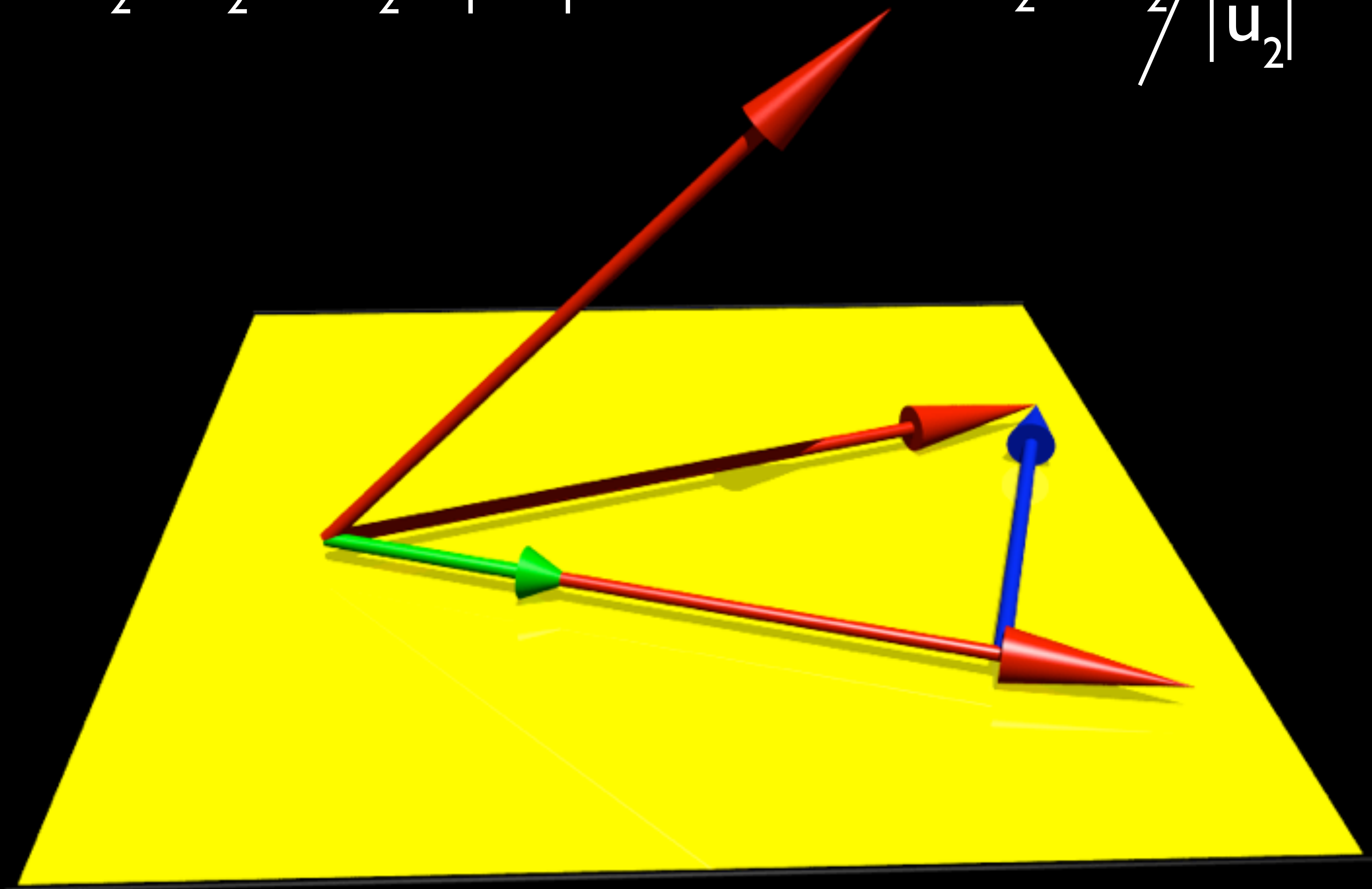
Gram Schmidt
QR Decomposition

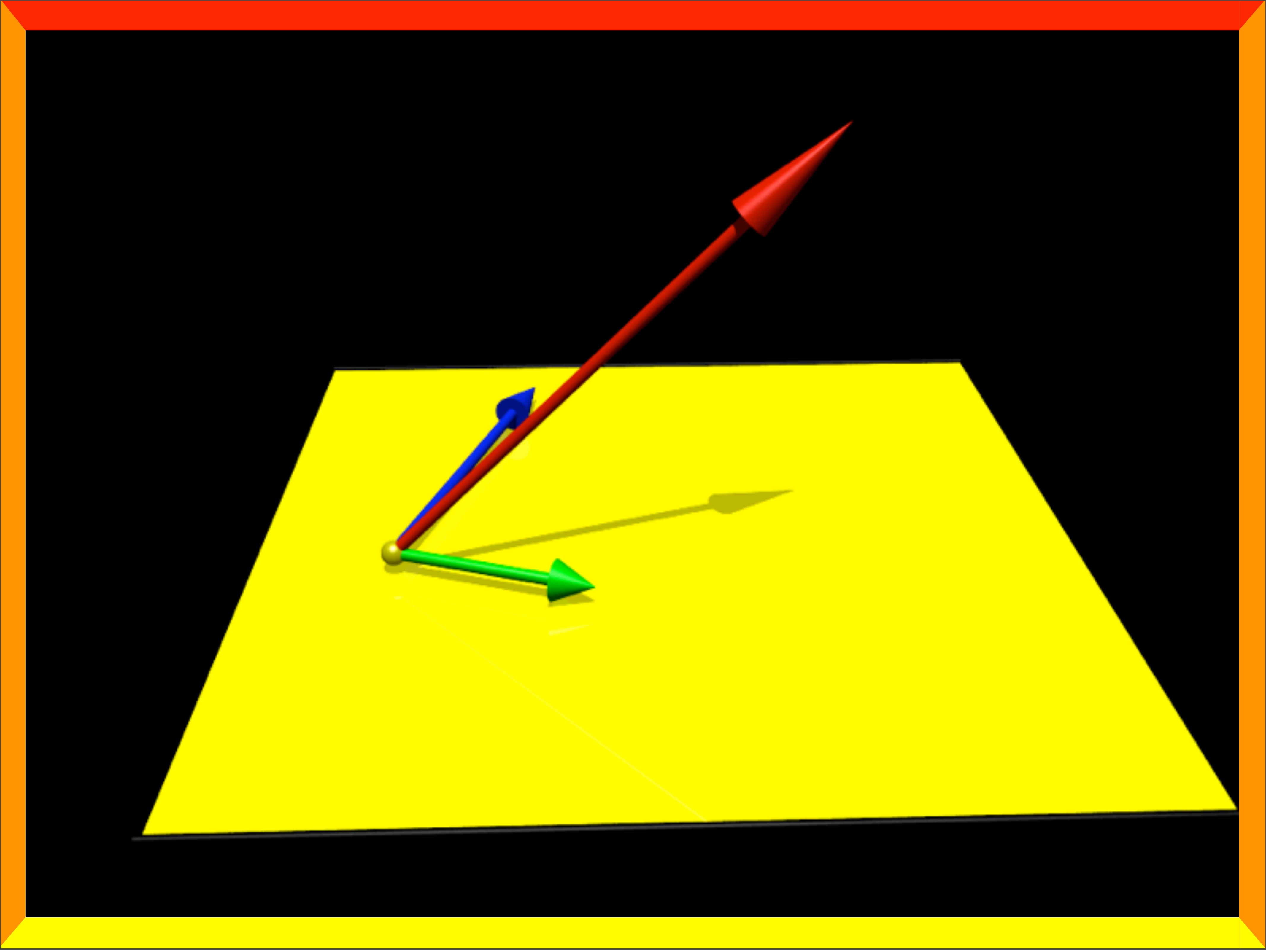


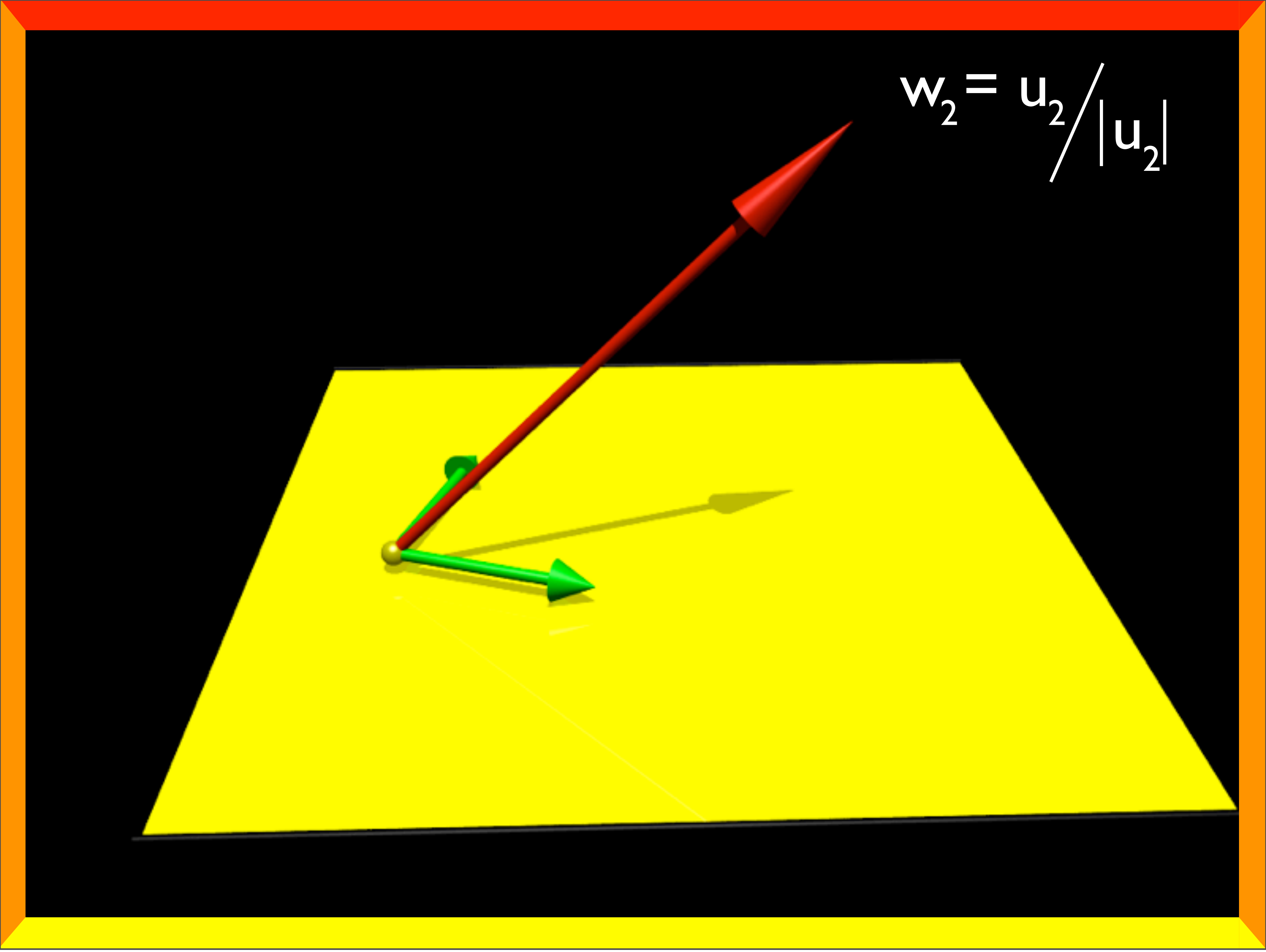

$u_1 = v_1$
$w_1 = u_1 / |u_1|$

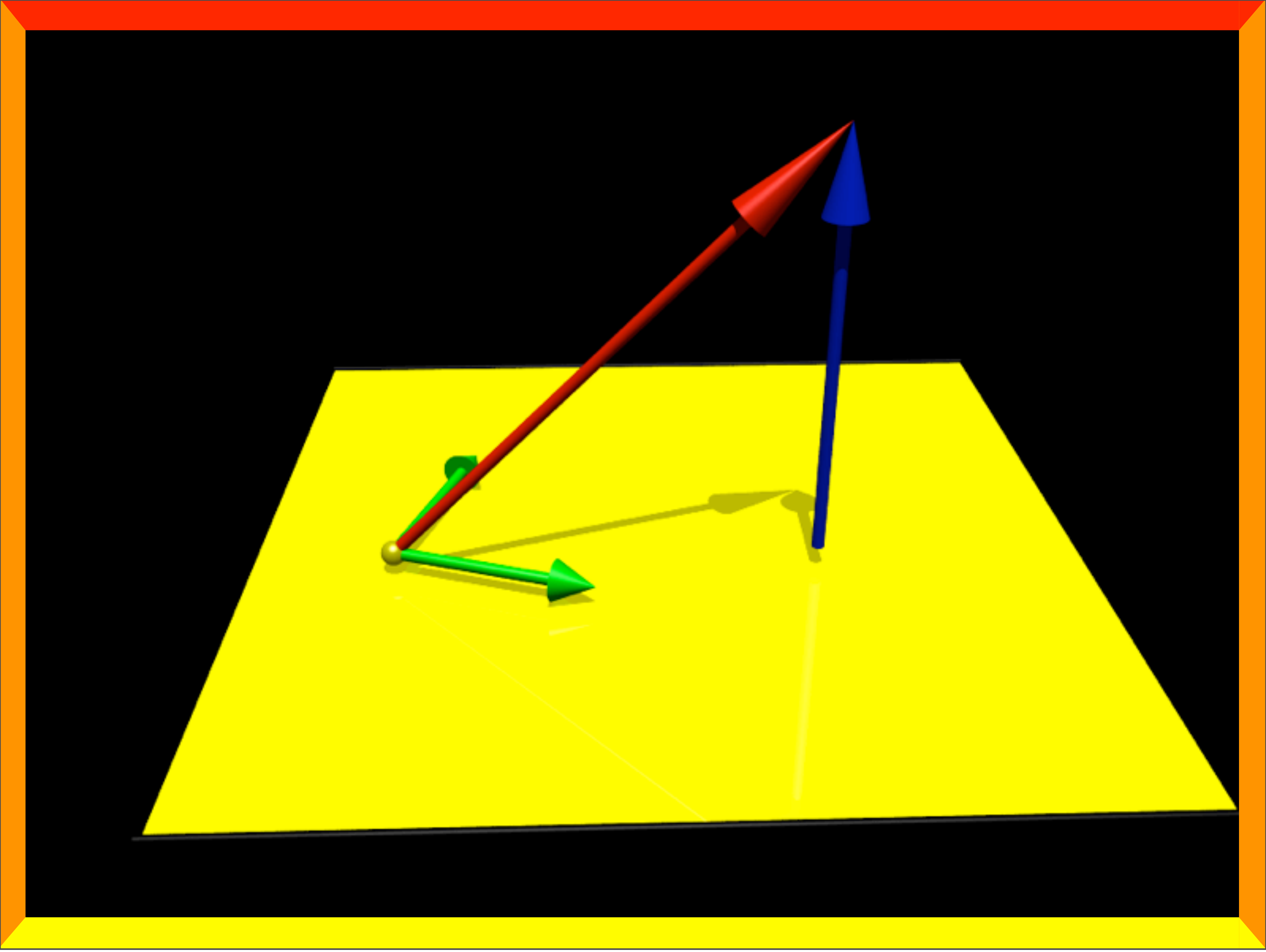
$$u_2 = v_2 - (v_2 . w_1)\, w_1$$

$$w_2 = u_2 / |u_2|$$



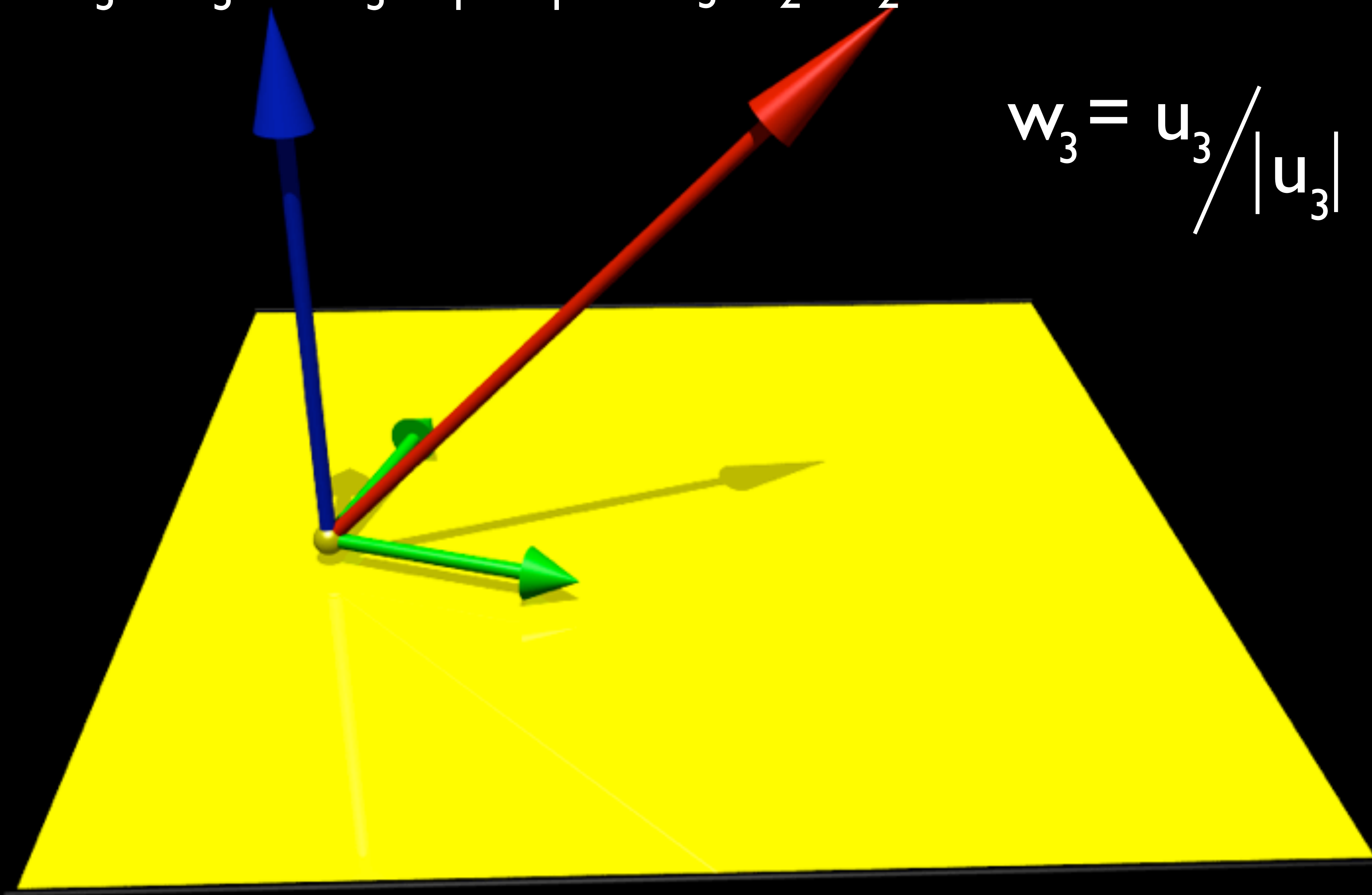



$w_2 = u_2 / |u_2|$

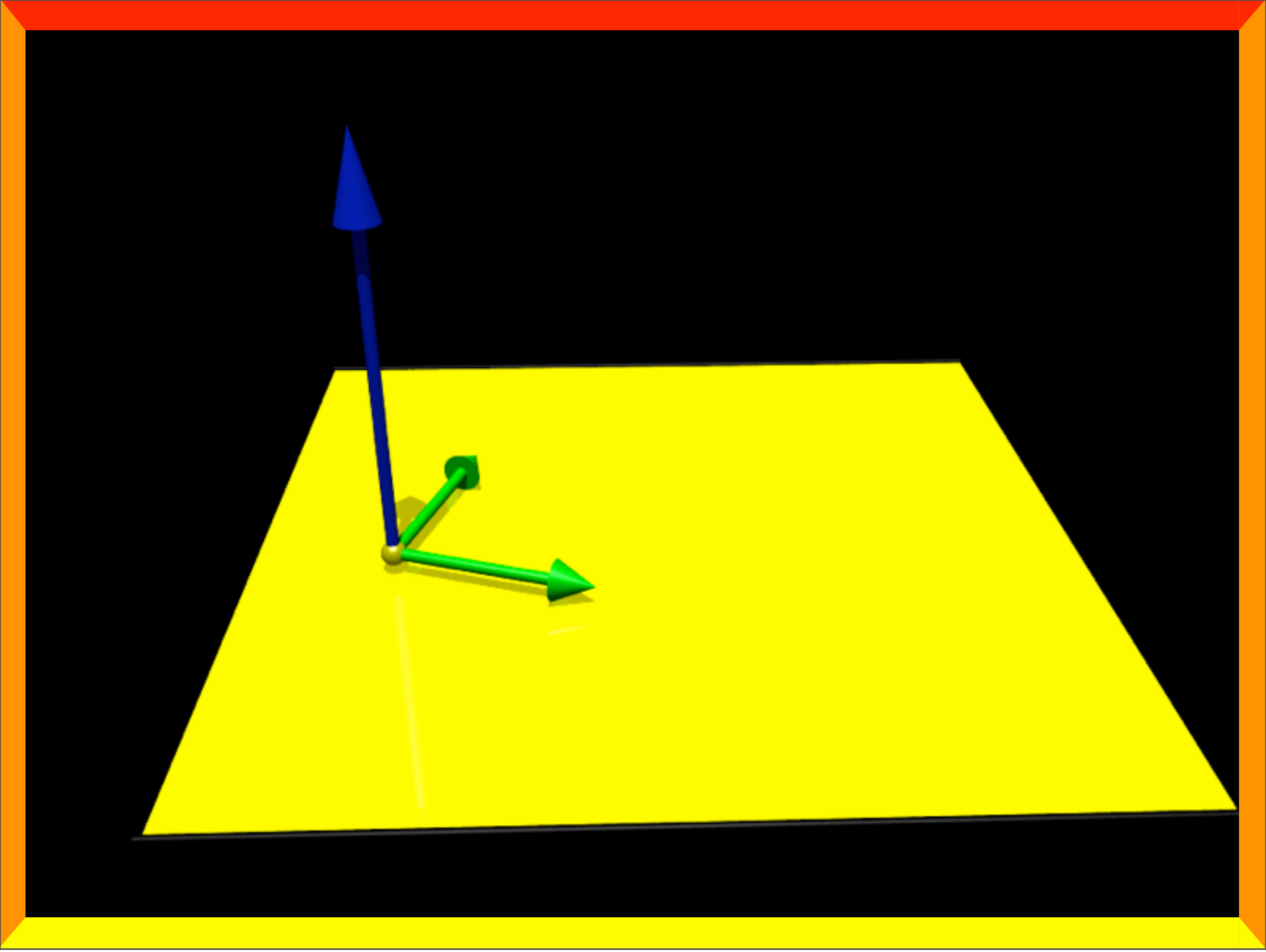


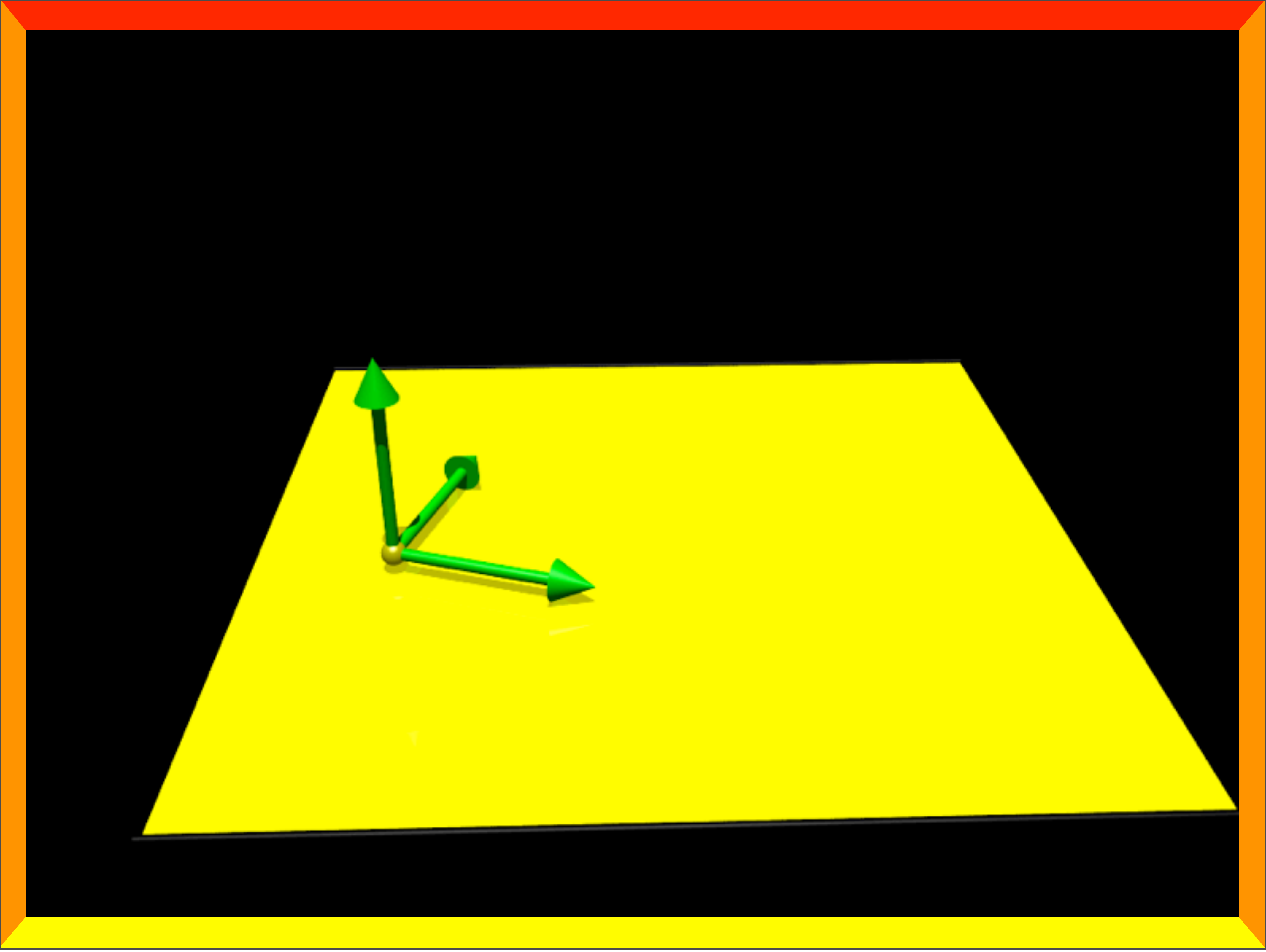
Monday, April 6, 2009

$$u_3 = v_3 - (v_3 \cdot w_1)\, w_1 - (v_3 \cdot w_2)\, w_2$$



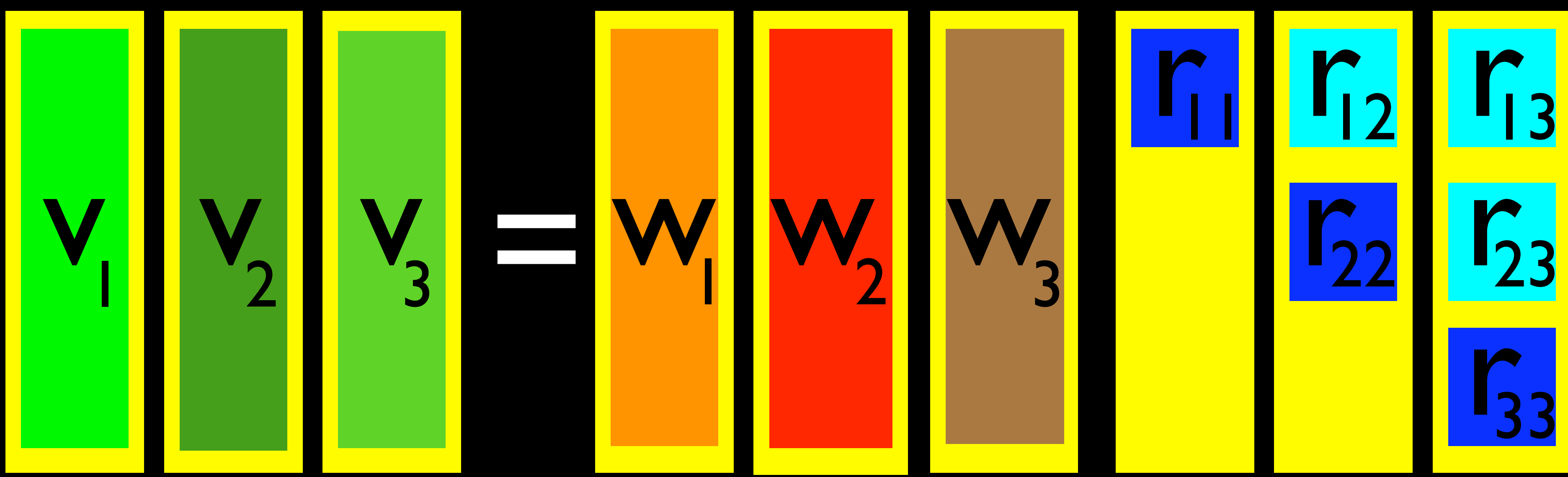
$$w_3 = u_3 / |u_3|$$





$$A = Q\,R$$

$$r_{ii} = |u_i| \qquad r_{ij} = (w_i \cdot v_j)$$



blackboard problem

back to the projection problem:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Find the QR decomposition

Alternatively. Make QR decomposition $A = QR$ and form $P = Q^TQ$ with

$$Q = \begin{bmatrix} 0 & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{6} \end{bmatrix}$$


QR decompositions can be painful

Determinants

$$\det\begin{pmatrix} a \end{pmatrix} = a$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$$

$$+ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$aei + bfg + cdh \ - \ bdi - ceg - ahf$$

after searching thousands and thousands of anagrams, I found the following memnonic device:

$$\det\begin{pmatrix} a & e & s \\ o & i & u \\ d & t & m \end{pmatrix} =$$

mia + deu + sot

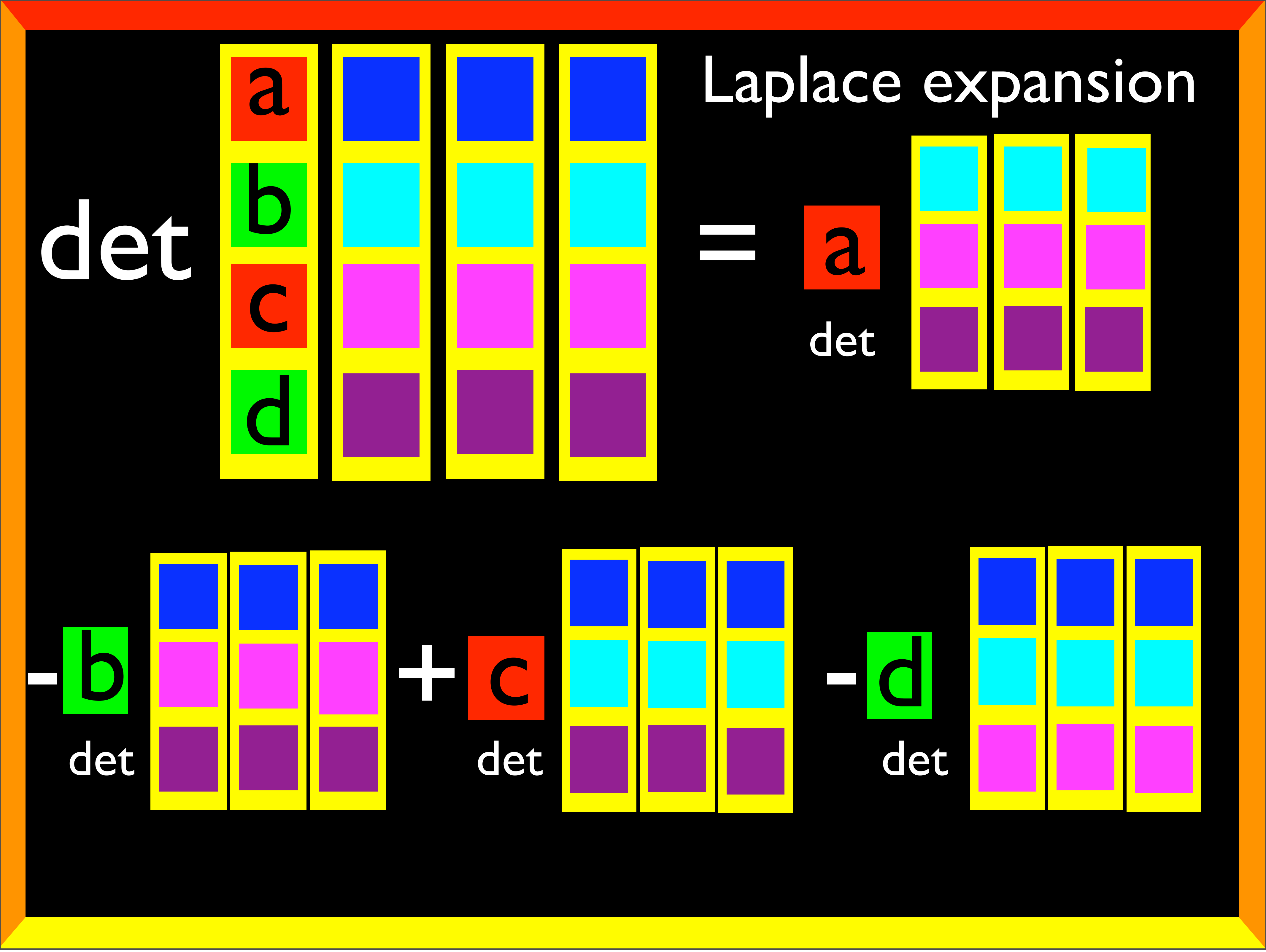
-tua - dis -emo

$$\det\begin{pmatrix} a & e & s \\ o & i & u \\ d & t & m \end{pmatrix} =$$

mia + deu + sot

-tua - dis -emo

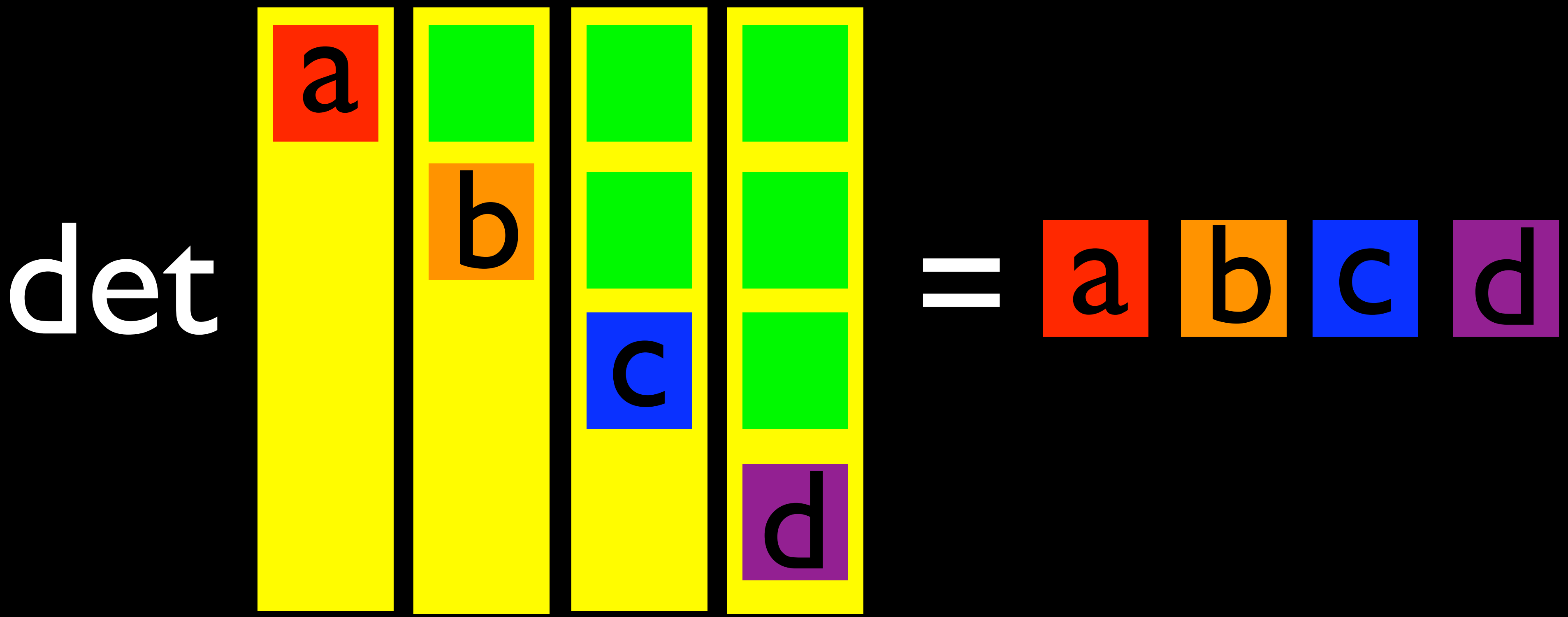

the composer


Laplace expansion
det
a
b
c
d
=
a
det
-
b
det
+
c
det
-
d
det

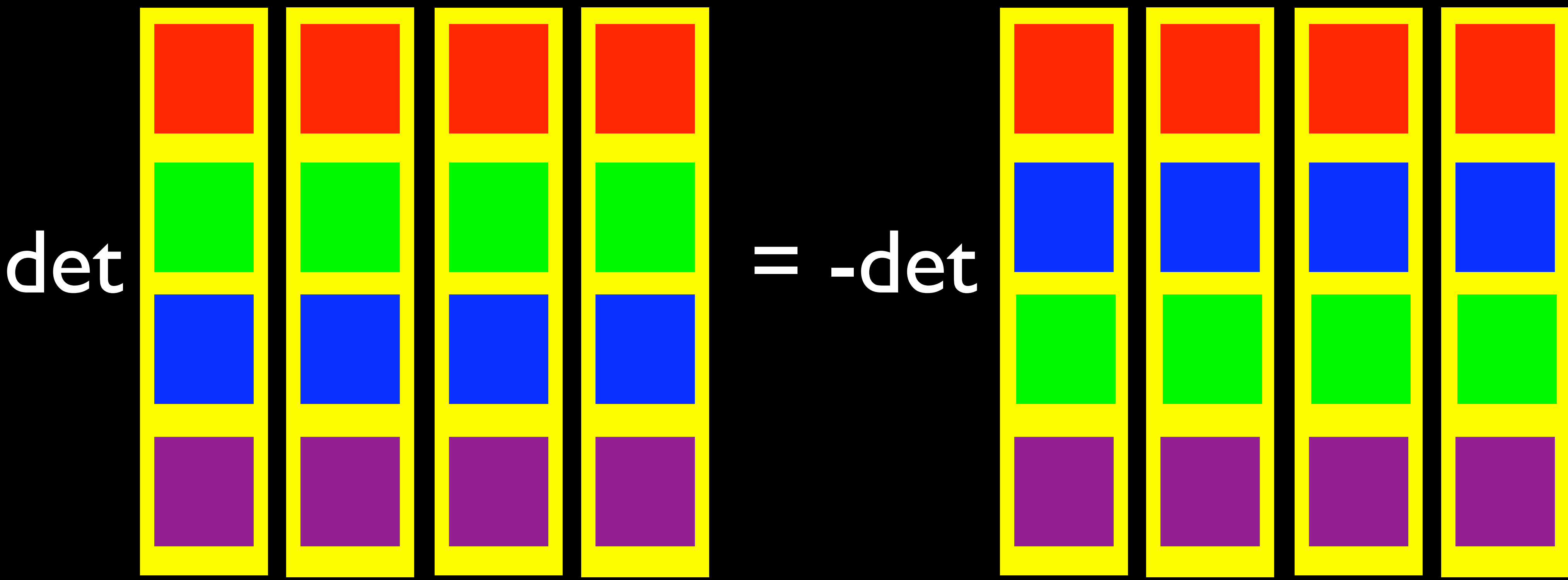
a handy trick.

$$\det\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det(A)\det(B)$$

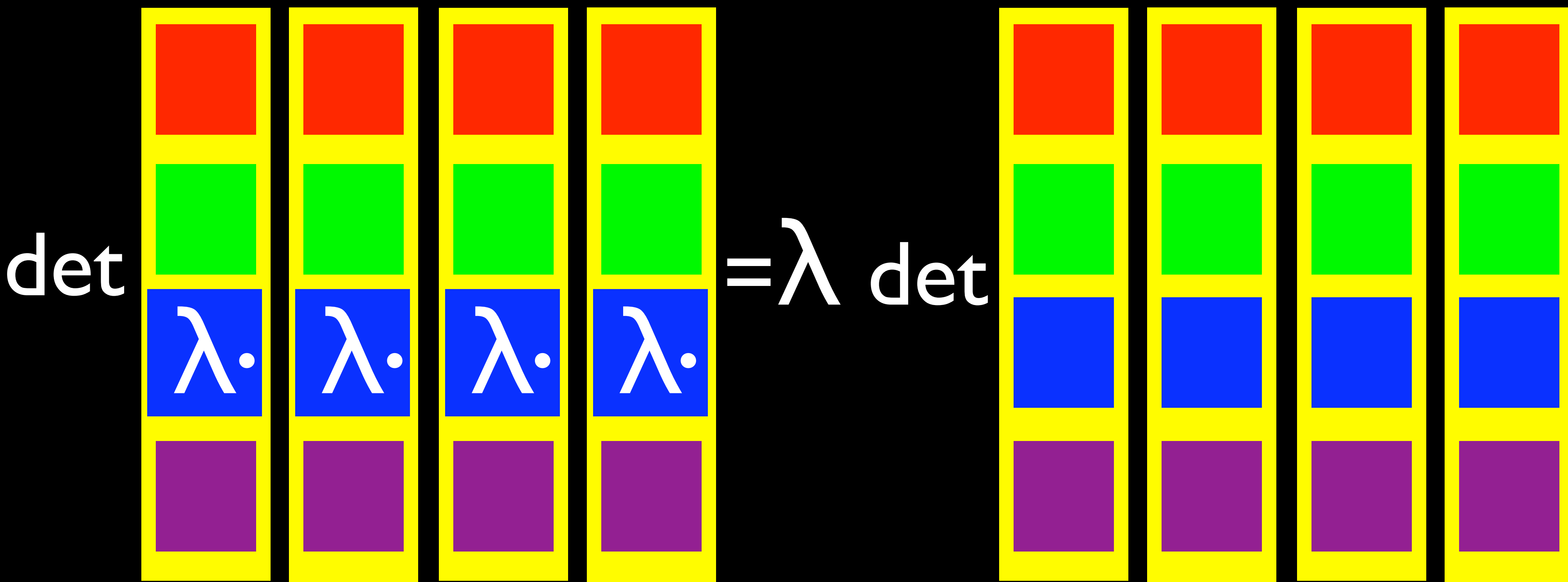
upper or lower triangular matrices



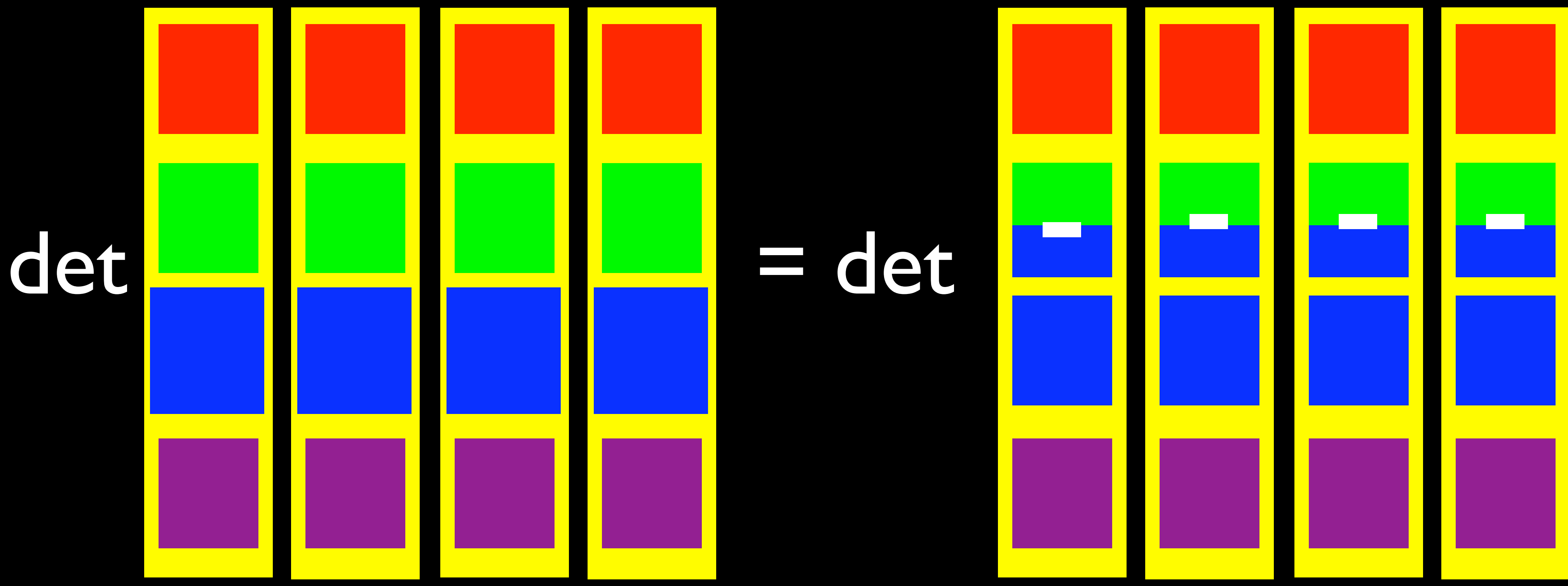
row reduction: swap



row reduction: scale



row reduction: subtract



blackboard problem

What is the determinant of

2	0	3	4	5	2
1	2	3	3	4	3
4	0	2	2	3	4
0	0	0	3	1	1
0	0	0	2	2	1
0	0	0	1	2	1


J
A
and
deal

ready?
the next slide
gives the
problem

What is the determinant of

$$\begin{vmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & -3 & 0 & 0 & 0 \\ 4 & 3 & 3 & 1 & 0 & 0 \\ 4 & 3 & 3 & 3 & -5 & 0 \\ 3 & 2 & 2 & 2 & 2 & 7 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Eigenvalues

$A\,v = \lambda v$

Examples:

Projections: eigenvalues 0 and 1

Rotations eigenvalues 1, -1 or complex

Reflections: eigenvalues -1 or 1

Shear: eigenvalue 1

The eigenvalues are the roots of the characteristic polynomial

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$f(\lambda) = \lambda^2 - tr(A)\lambda + det(A)$$

$$\lambda_1 = \frac{tr(A) + \sqrt{tr(A)^2 - 4det(A)}}{2}$$

$$\lambda_2 = \frac{tr(A) - \sqrt{tr(A)^2 - 4det(A)}}{2}$$

algebraic multiplicity

number of simultaneous roots of λ.

geometric multiplicity

dimension of ker(A-λI)

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

$$f_A(\lambda) = (\lambda - a)(\lambda - a)$$

algebraic multiplicity: 2

geometric multiplity: 1

Some good things to know

determinant

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

trace

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

determinant of power

$$\det(A^k) = \lambda_1^k \; \lambda_2^k \; \cdots \; \lambda_n^k$$

trace of power

$$\mathrm{tr}(A^k) = \lambda_1^k + \lambda_2^k + \ldots + \lambda_n^k$$

seeing eigenvalues this can be useful:

$$\begin{bmatrix} 11 & 12 & 0 \\ 13 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

blackboard problem

Find the eigenvalues and determinant of

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Eigenvectors

$Av=\lambda v$

$(A-\lambda I)v=0$

v is in the kernel of $A-\lambda I$

The eigenspace is a kernel. If all eigenvalues are different, we have one eigenvector for each eigenvalue.

The beginning is simple, almost comic.

compute kernels!

A real case

$$A = \begin{pmatrix} 11 & 12 \\ 20 & 3 \end{pmatrix}$$

$$\lambda_+ = 23 \qquad v_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_- = -9 \qquad v_- = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\lambda_+ = a+ib \qquad v_+ = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\lambda_- = a-ib \qquad v_- = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Example: (regular transition matrix)

$$A = \begin{pmatrix} 1/2 & 1/5 \\ 1/2 & 4/5 \end{pmatrix}$$

$$\lambda_+ = 1$$

A^T has eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\lambda_- = 3/10$$ from trace

A^{100} has eigenvalues 1 and $(3/10)^{100}$

blackboard problem

Find all the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 5 \end{bmatrix}$$

Find all the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Watch out for zero eigenvalues.

Discrete dynamical systems

$$x(t+1) = A\,x(t)$$

has solution:

$$x(n) = A^n x(0)$$

but this does not give any insight.

We can find
closed form solutions.

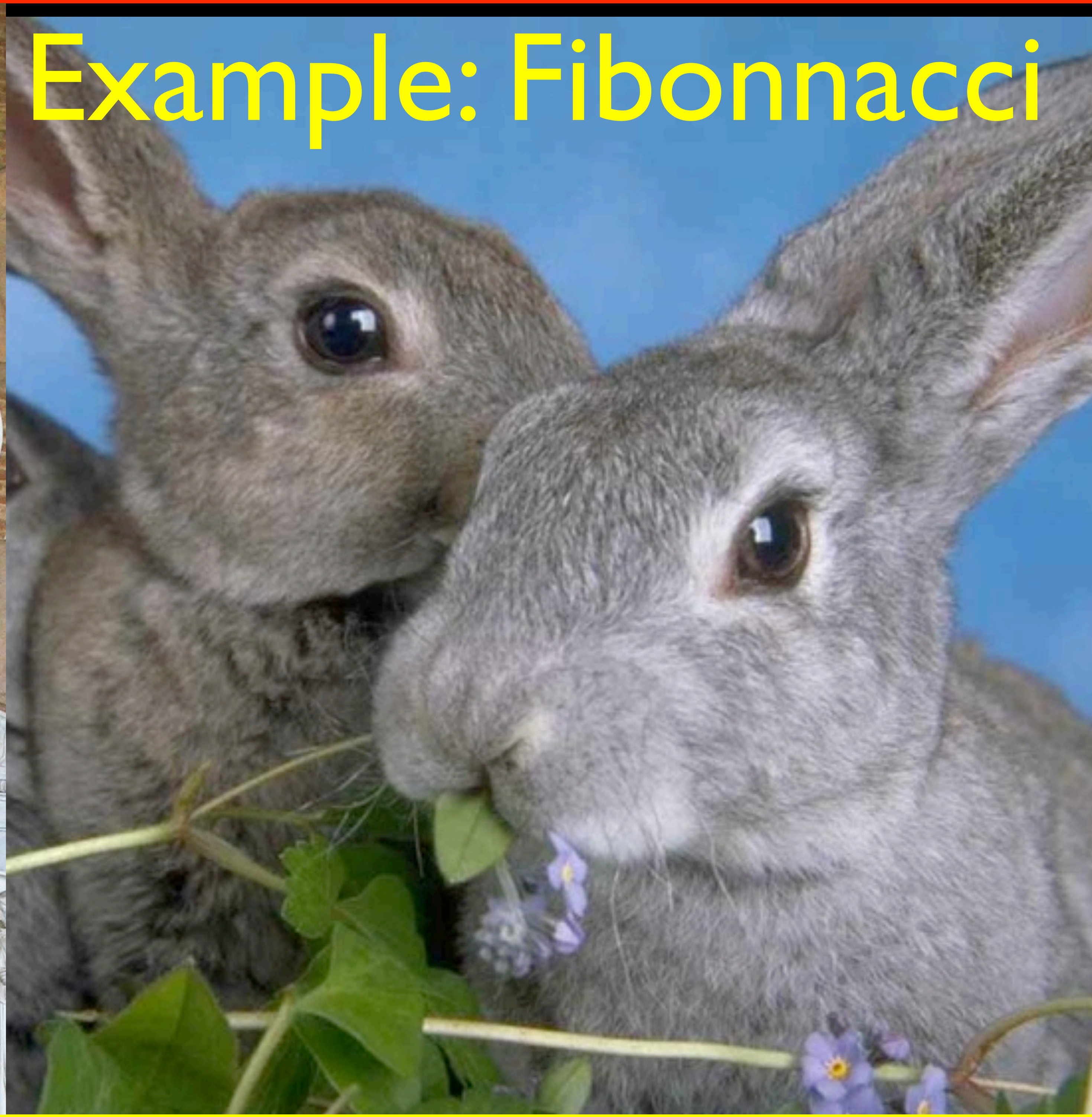
$$x(0) = c_1 v_1 + \ldots + c_n v_n$$

$$A v_k = \lambda_k v_k$$

$$x(t) = c_1 \lambda_1^t v_1 + \ldots + c_n \lambda_n^t v_n$$

closed form solution

Example: Fibonnacci



$$x(n+1) = x(n) + x(n-1) \qquad 1,1,2,3,5,8,13,21,\ldots$$

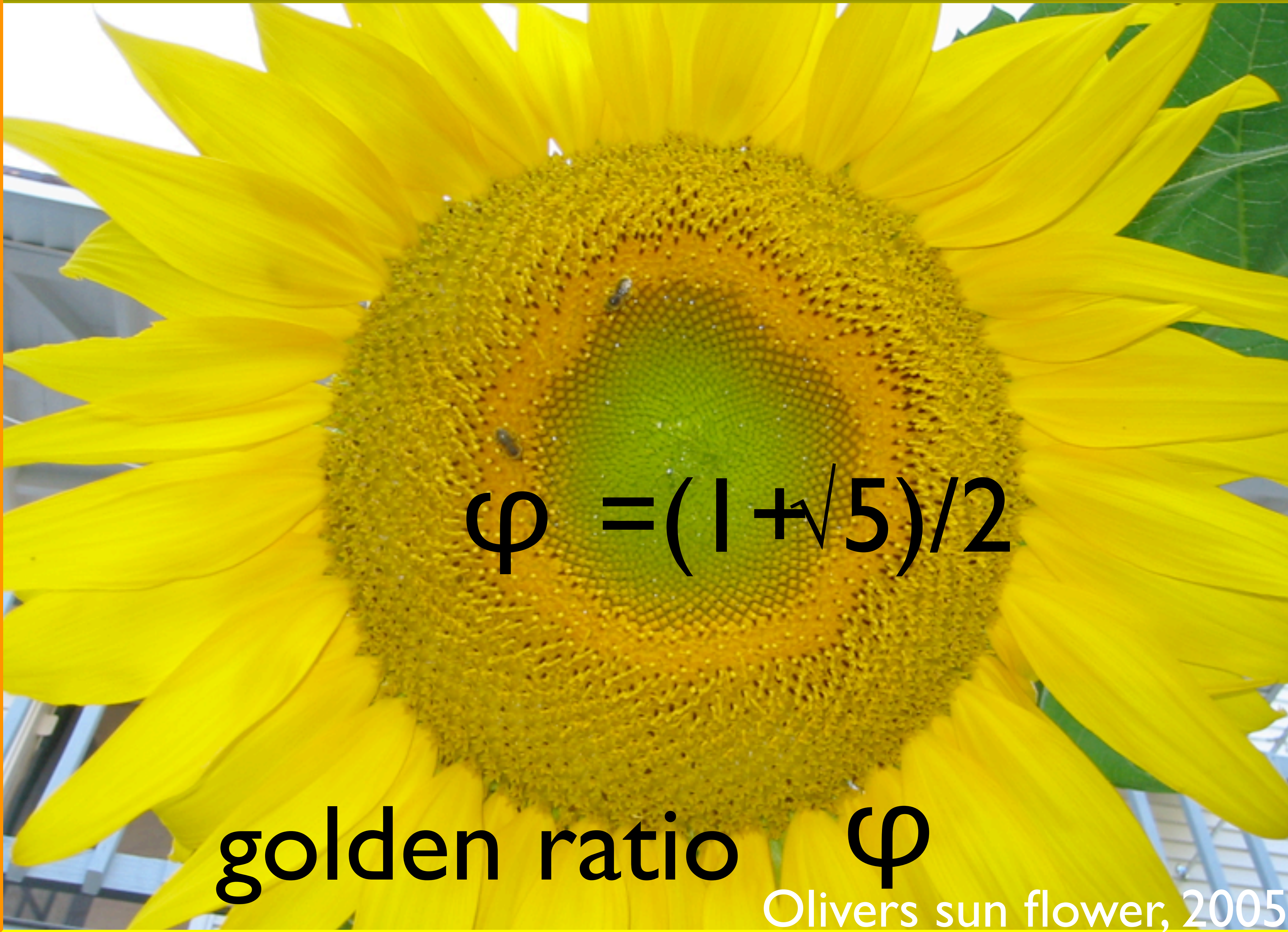
$$\begin{pmatrix} x(n+1) \\ x(n) \end{pmatrix} = \begin{pmatrix} x(n)+x(n-1) \\ x(n) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(n) \\ x(n-1) \end{pmatrix}$$

characteristic polynomial: $\lambda^2 - \lambda - 1 = f_A(\lambda)$

eigenvalues=$\varphi, 1-\varphi$ eigenvectors= $\begin{pmatrix} \varphi \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1-\varphi \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \begin{pmatrix} 1-\varphi \\ 1 \end{pmatrix}$$

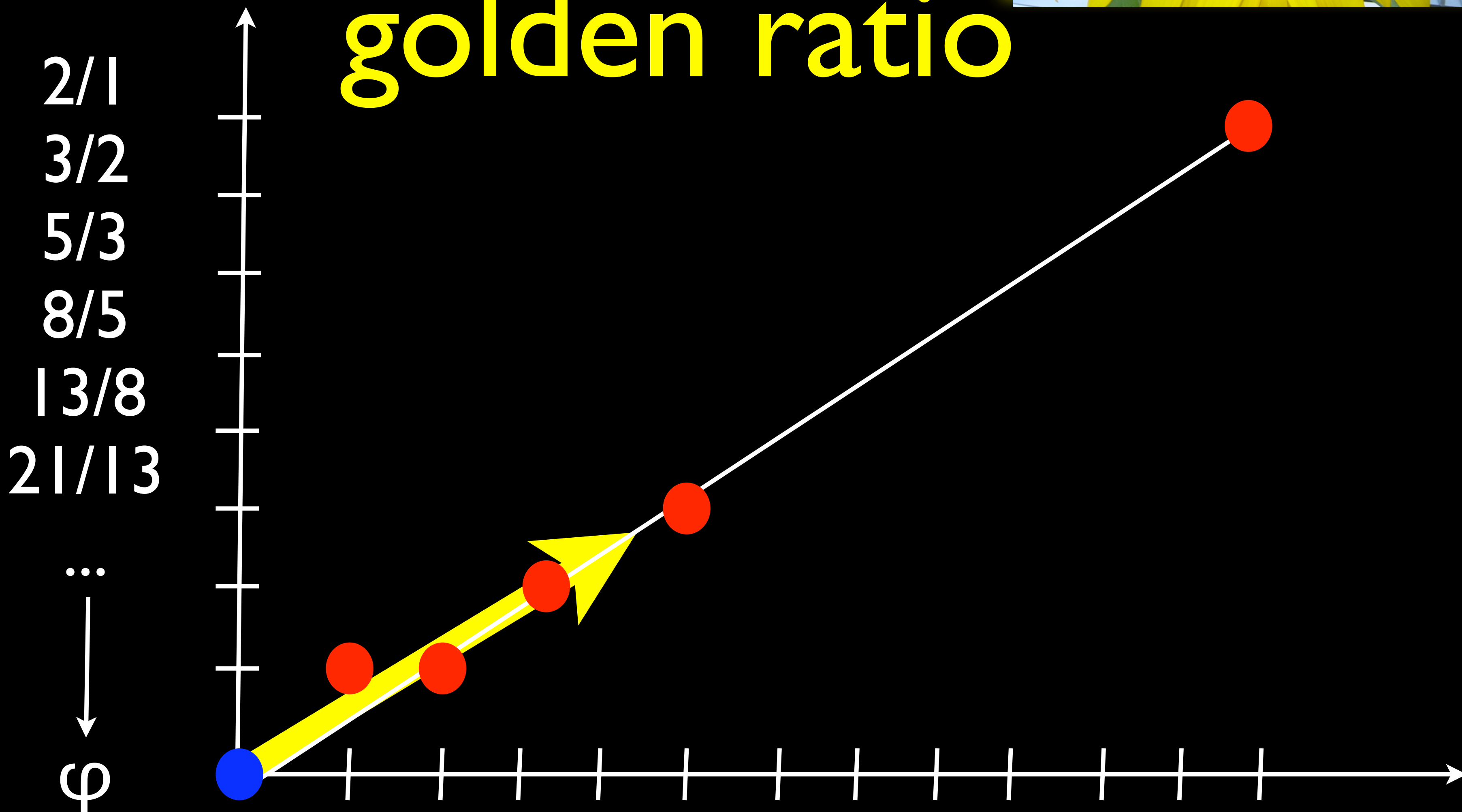
$$x(n) = (\varphi^n - (1-\varphi)^n)/\sqrt{5}$$


φ =(1+√5)/2
golden ratio φ
Olivers sun flower, 2005

$$\varphi = \frac{\sqrt{5}+1}{2} = 1.616...$$

$$1/\varphi = \varphi - 1 = 0.616...$$

golden ratio



blackboard problem

The Lilac Bush problem

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ new branches
old branches

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

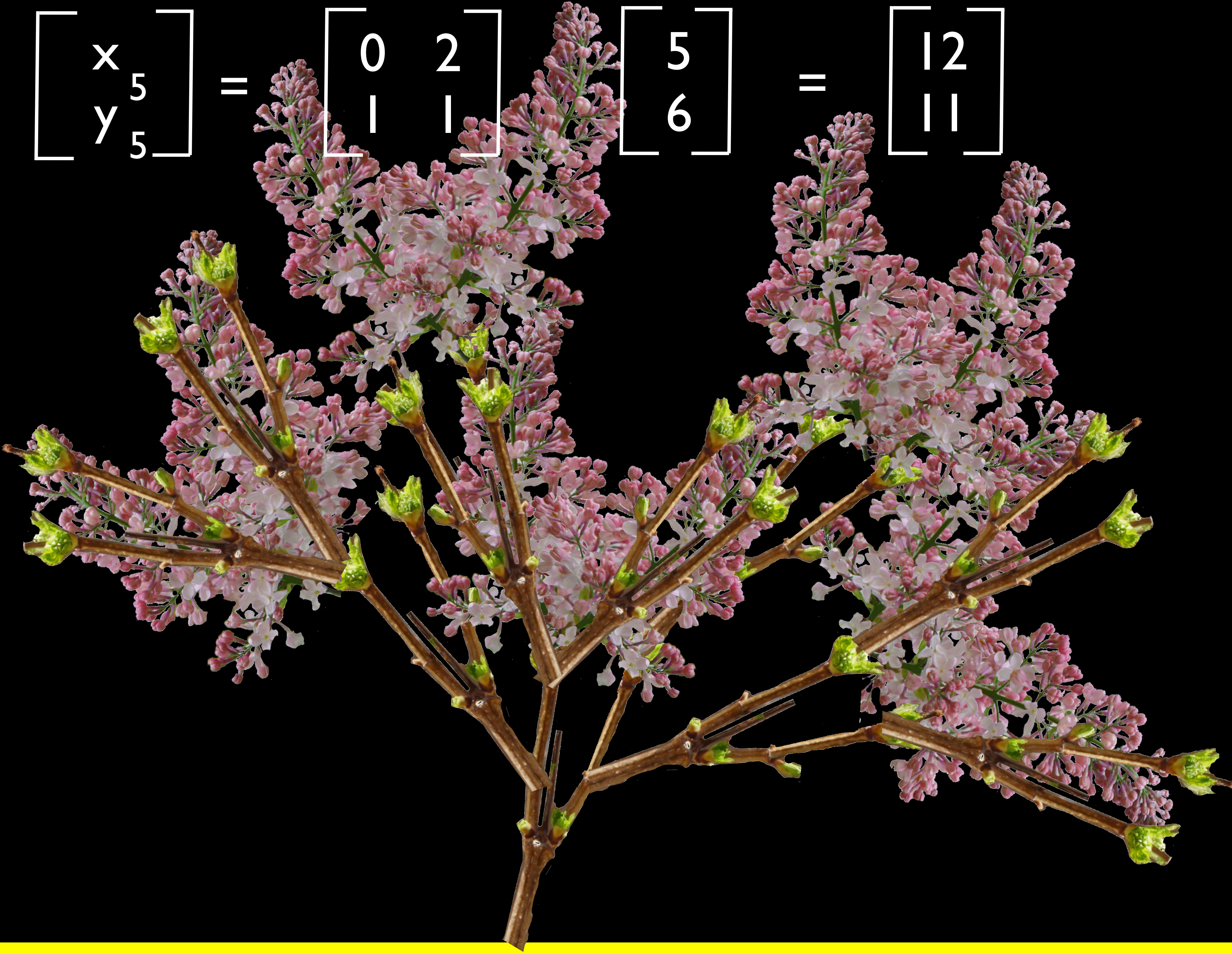
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

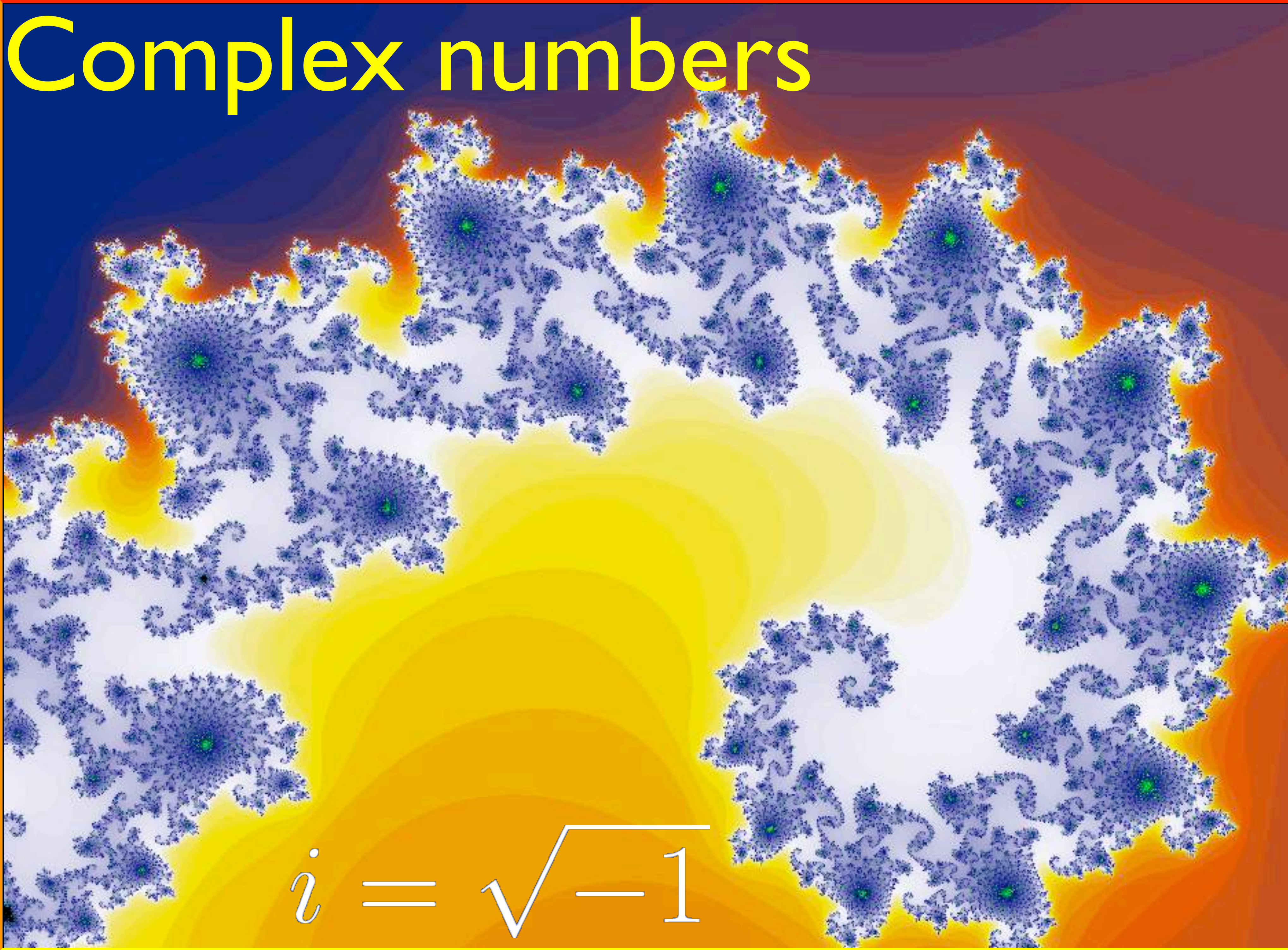
$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$$



Complex numbers



$$i = \sqrt{-1}$$

Complex numbers

$$i = \sqrt{-1}$$

Gauss in 1825 : "The true metaphysics of the square root of -1 is elusive".

Euler Formula

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

Is the gateway to most secrets in complex numbers.

Proof:

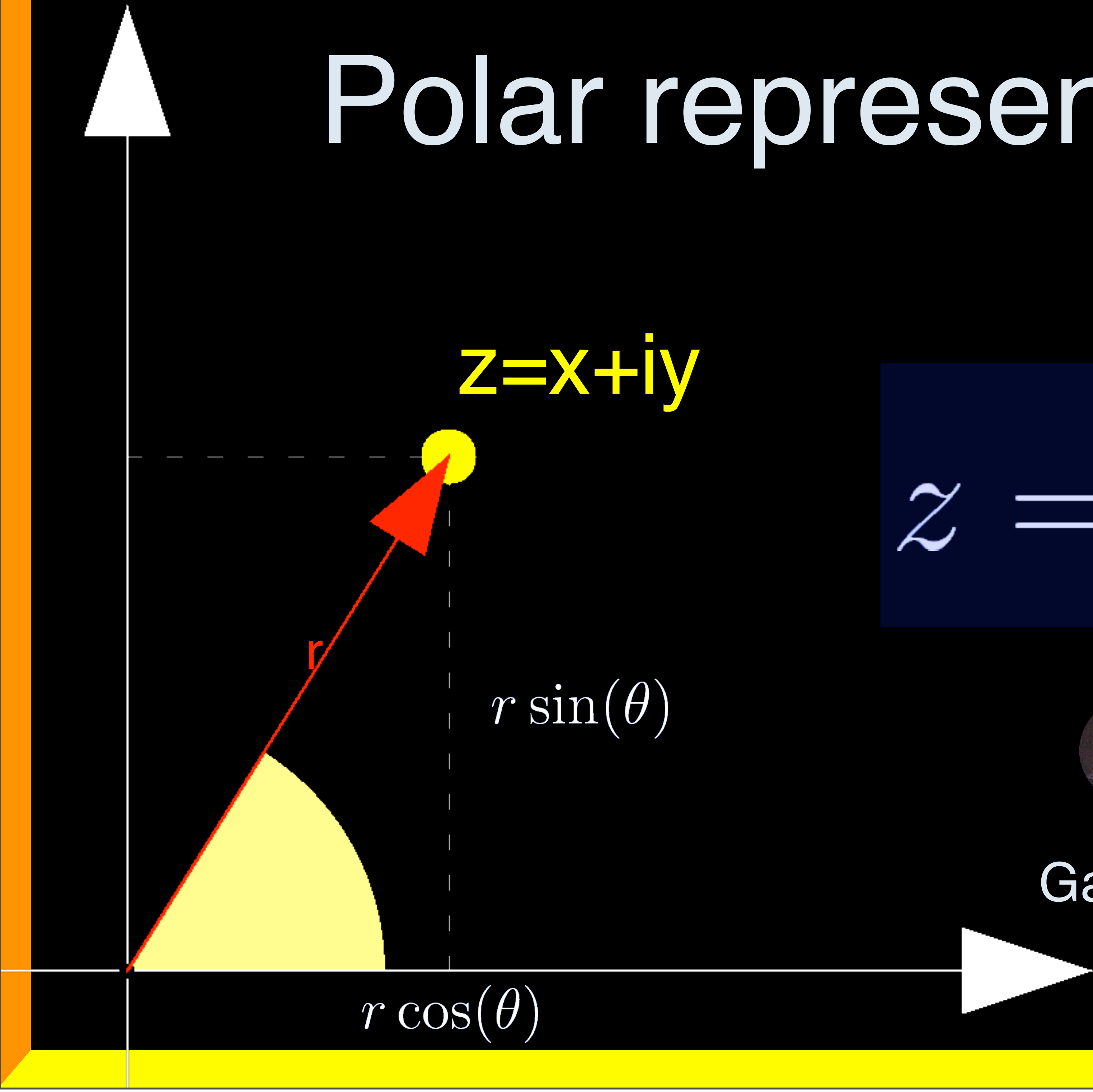
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

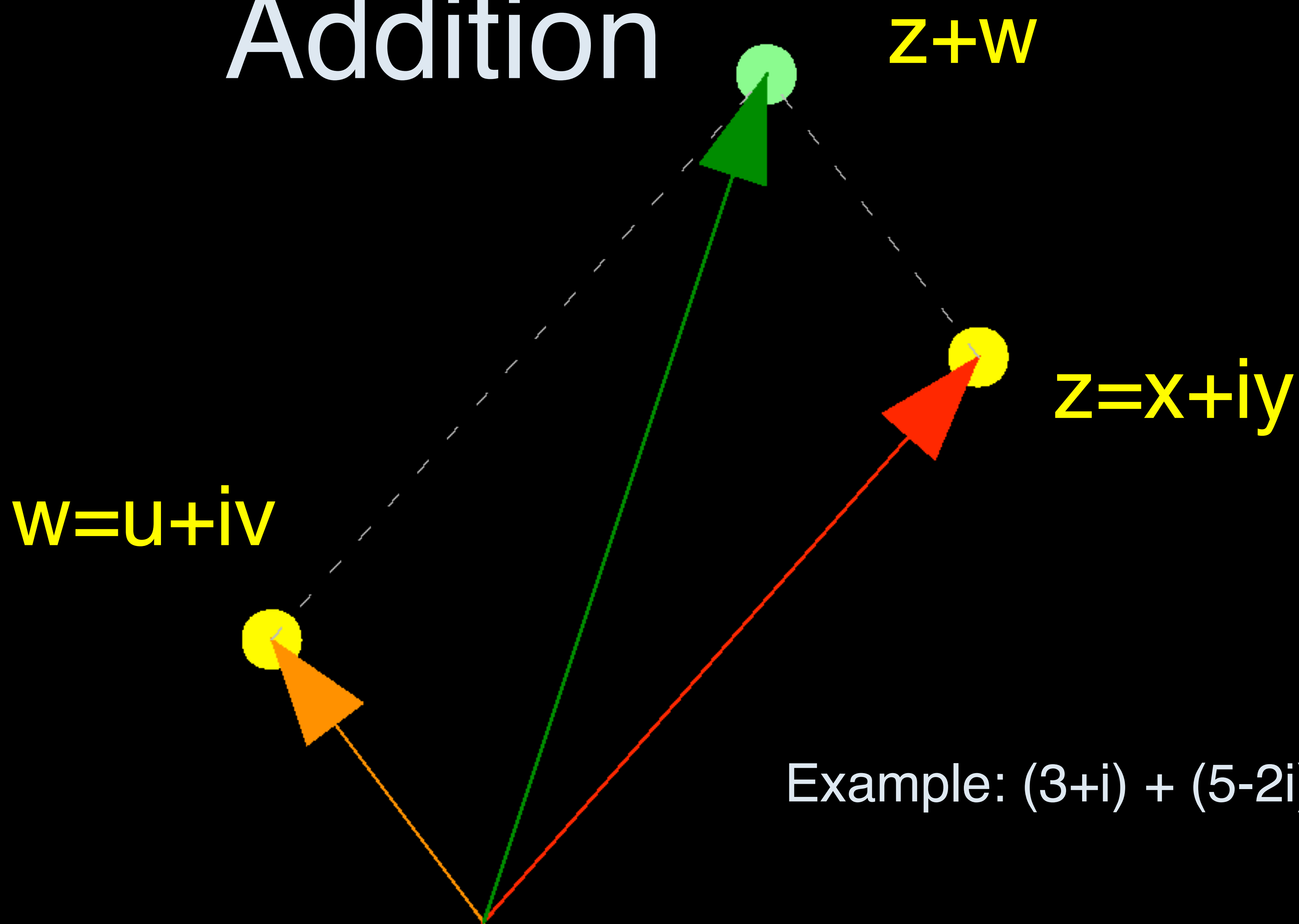
Polar representation



$$z = re^{i\theta}$$

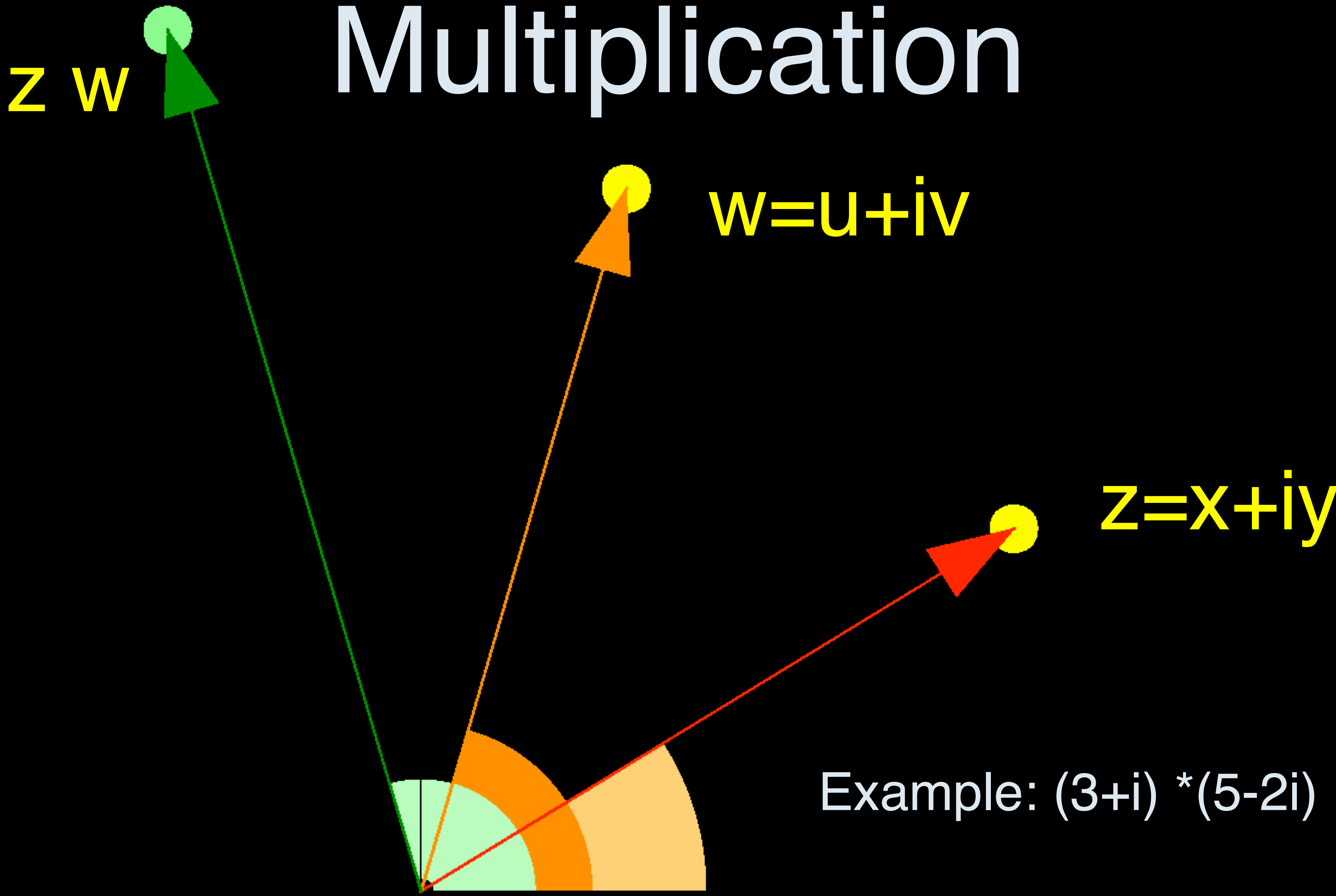
Gauss Plane

Addition

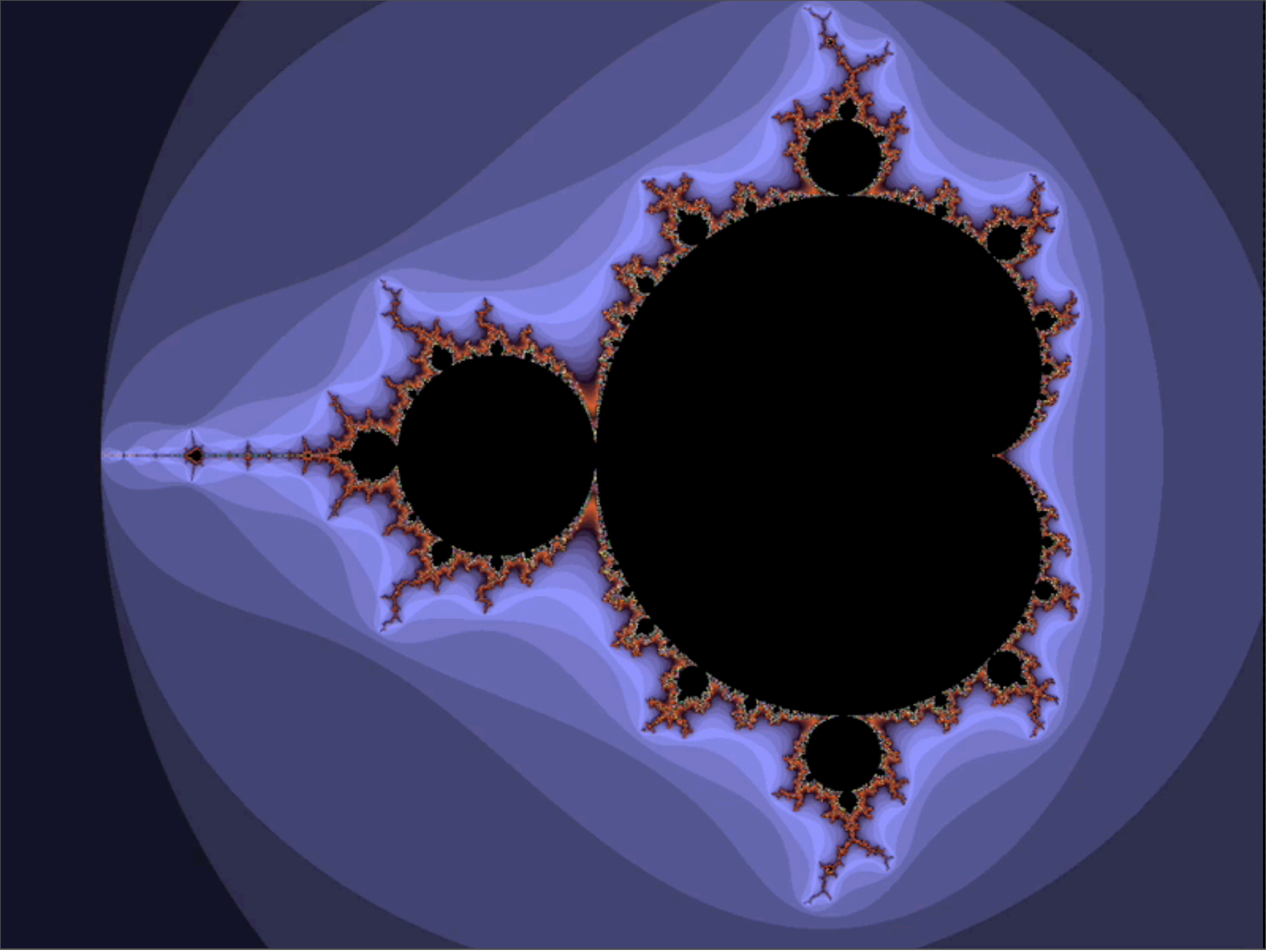


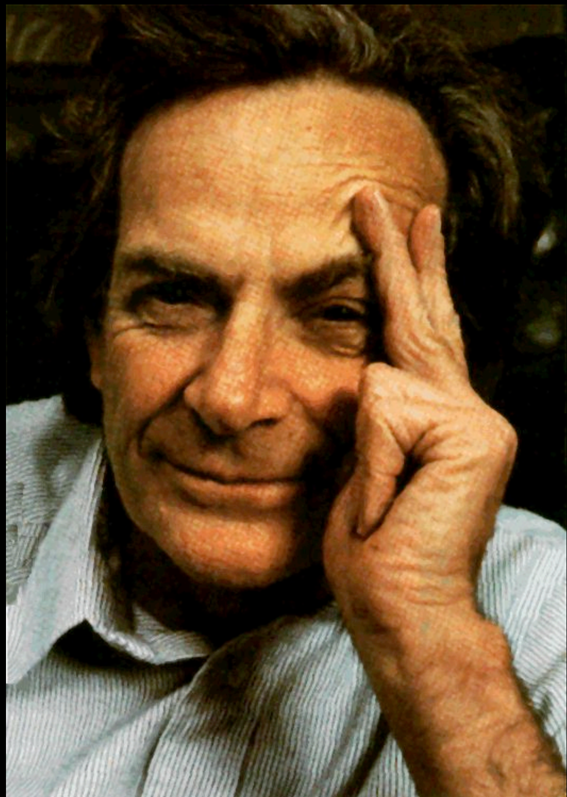
Example: (3+i) + (5-2i)

Multiplication



Example: (3+i) *(5-2i)

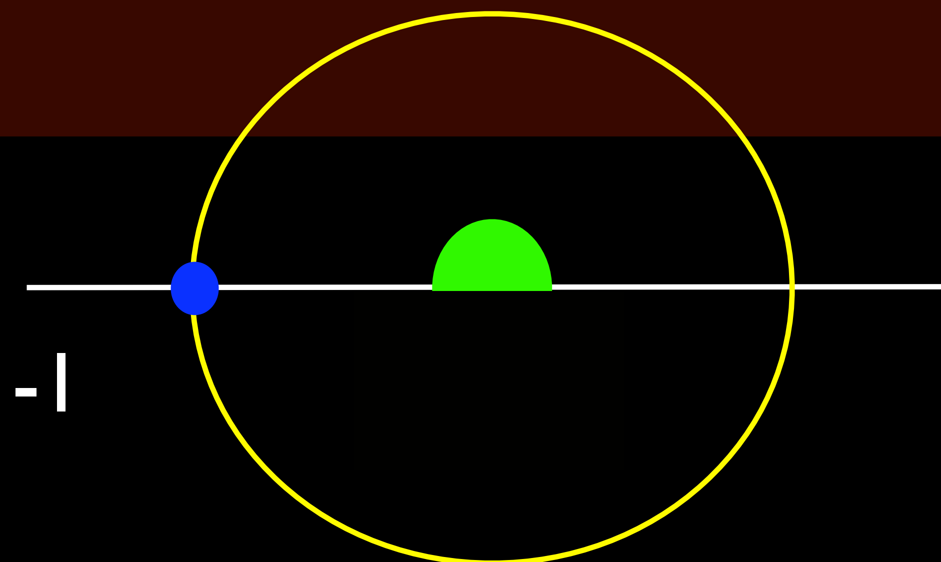




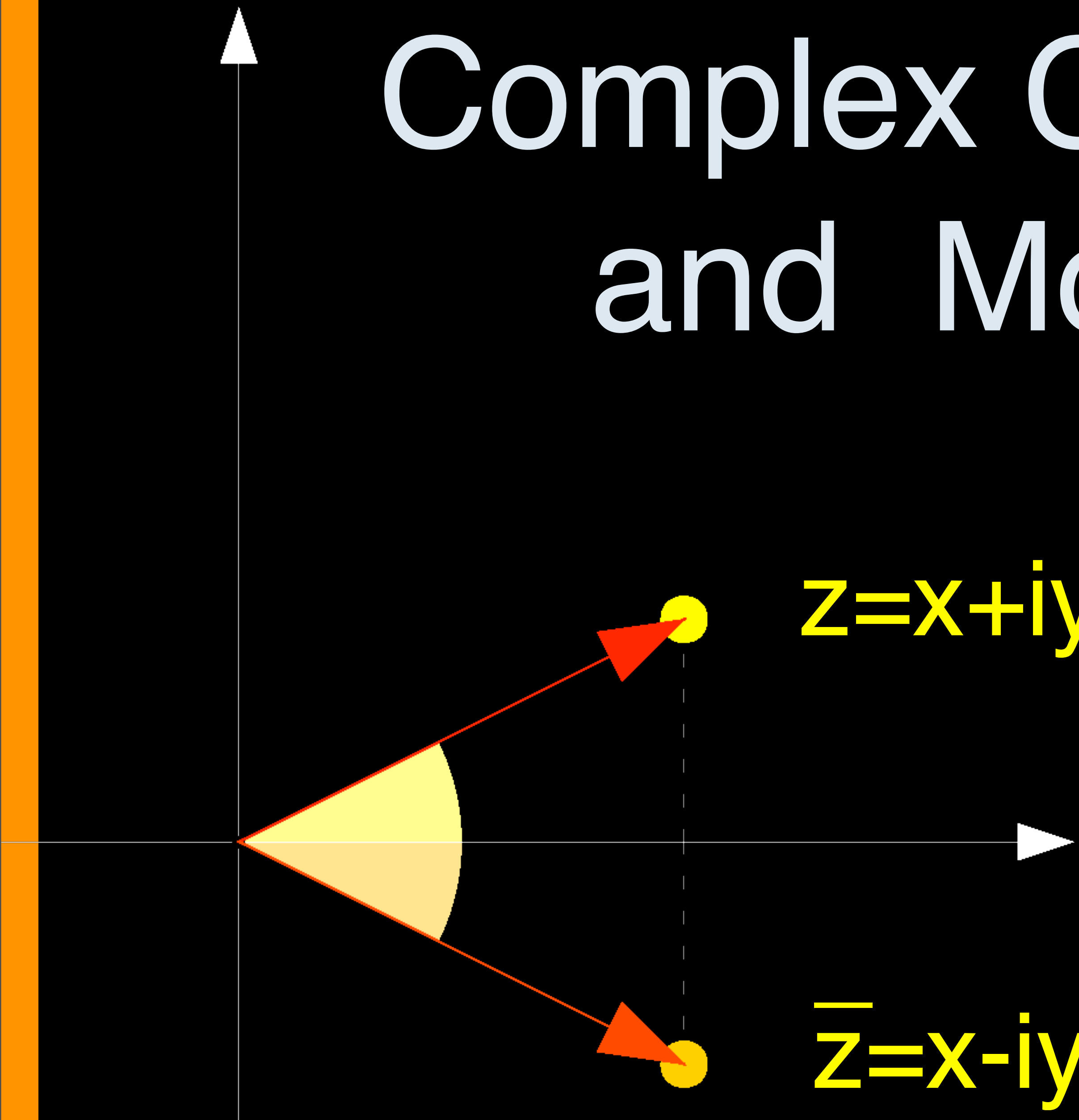
feynman

“The most remarkable formula in math”

$$1 + e^{i\pi} = 0$$



Complex Conjugate and Modulus

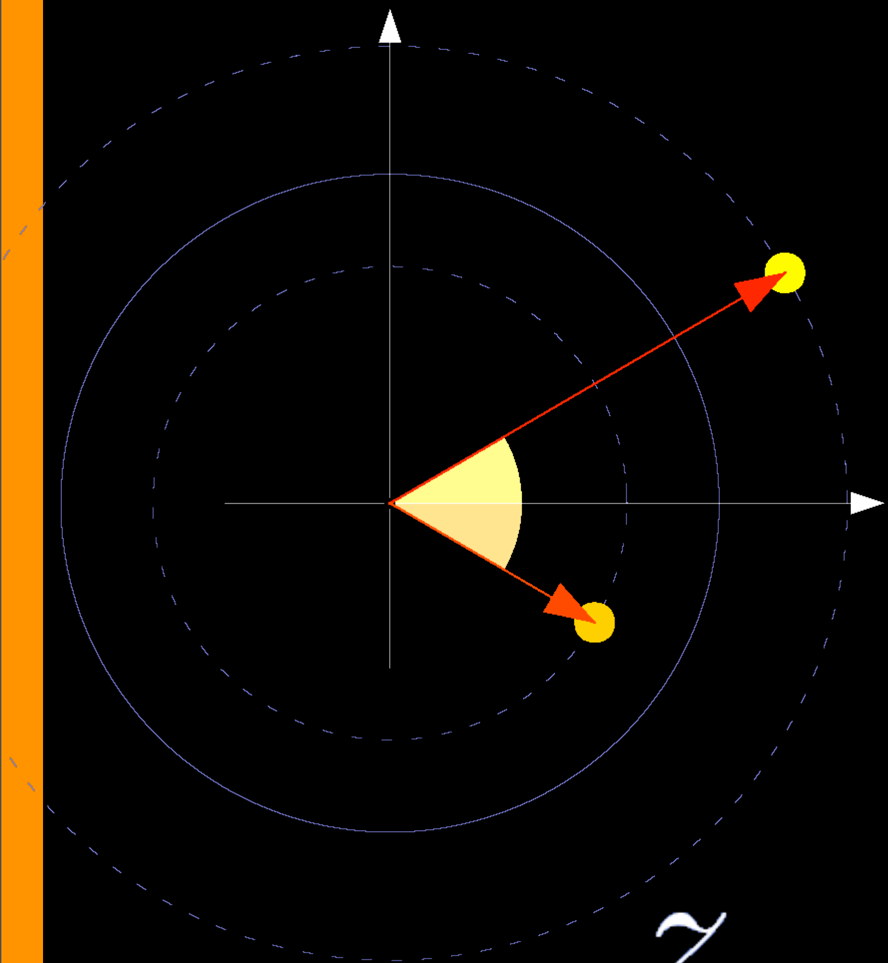


$z=x+iy$

$|z|^2=z\bar{z}$

$\bar{z}=x-iy$

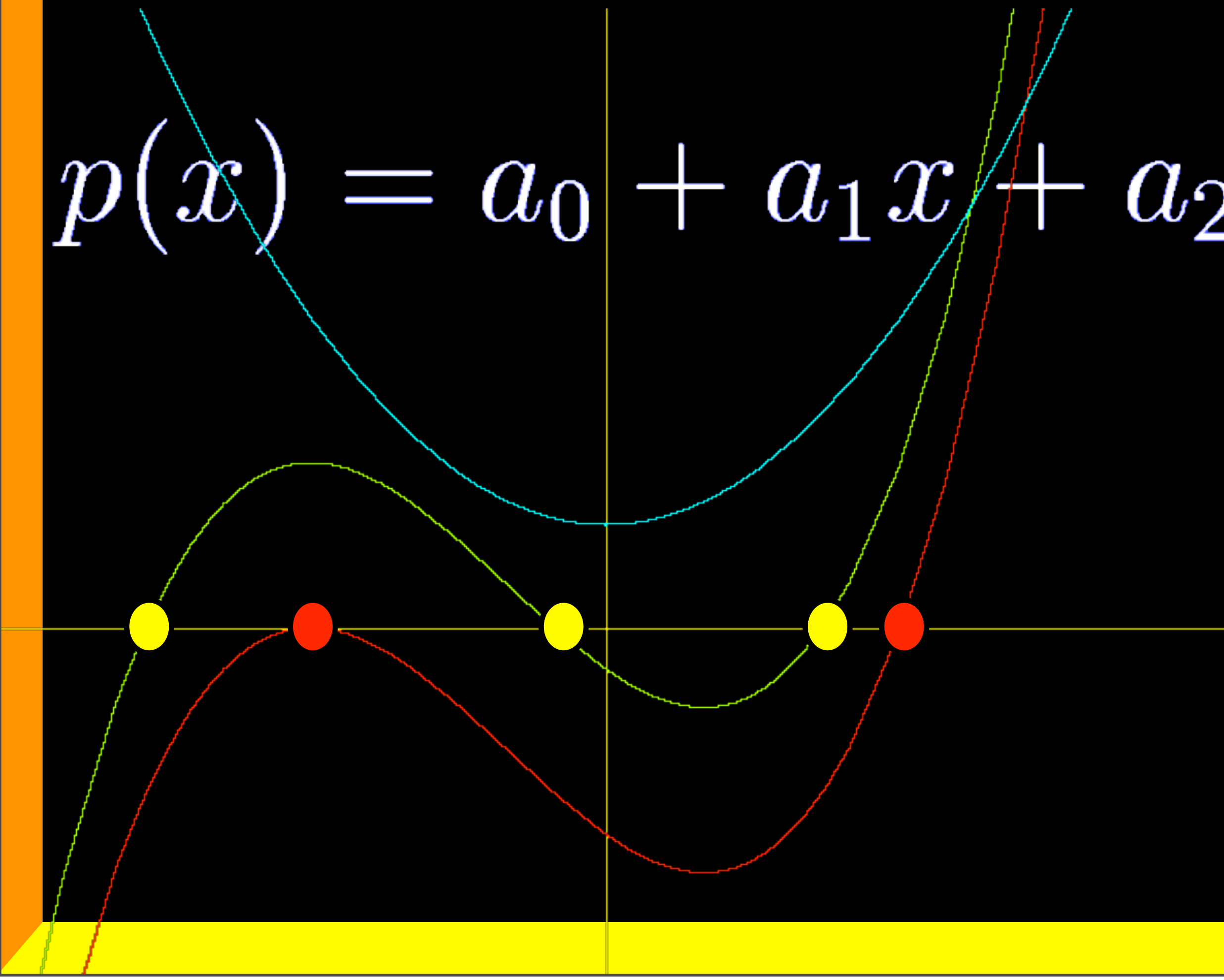
Division



$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

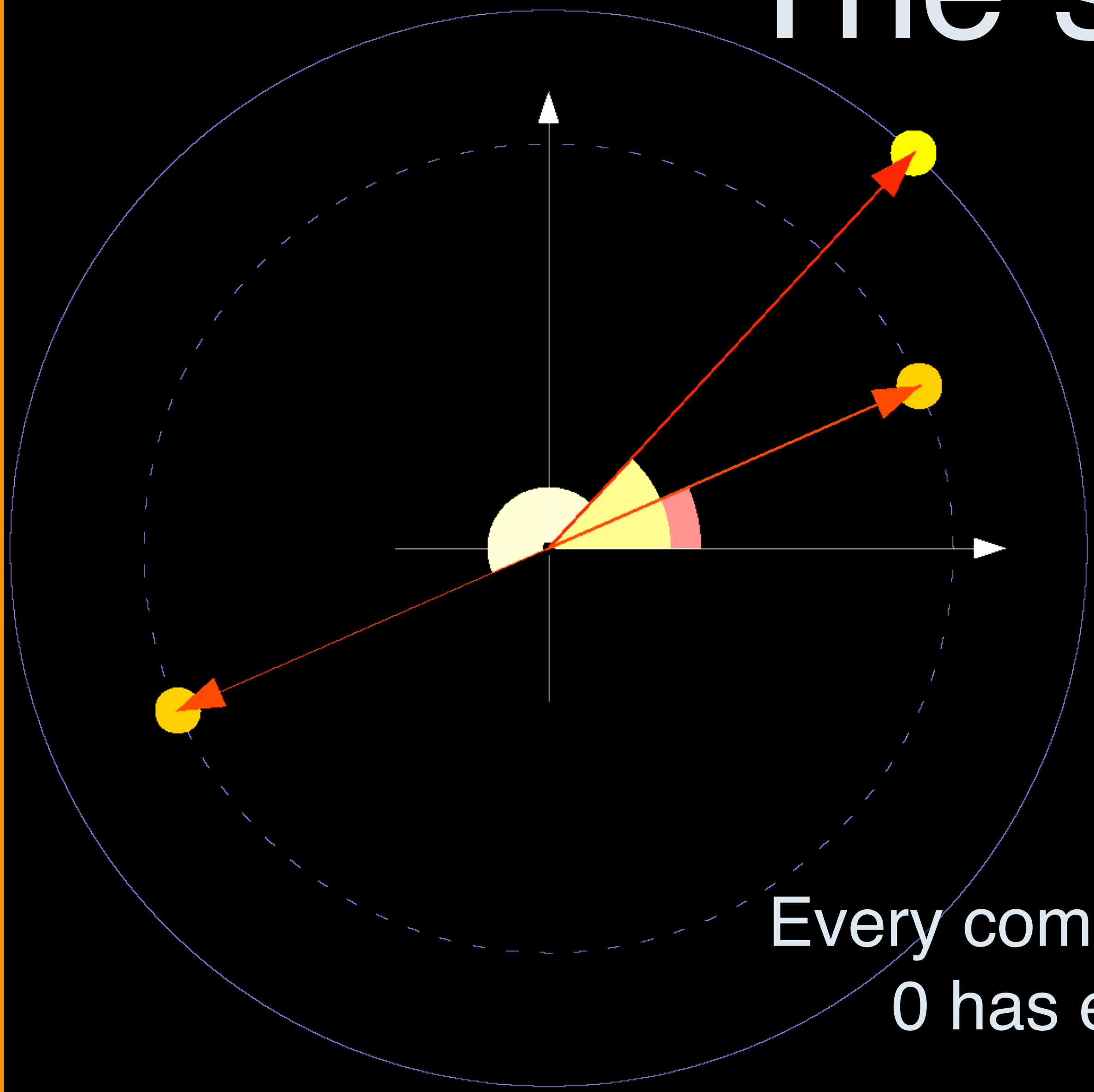
Fundamental theorem of algebra

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$



A polynomial of degree n has exactly n roots p(x)=0

The square root

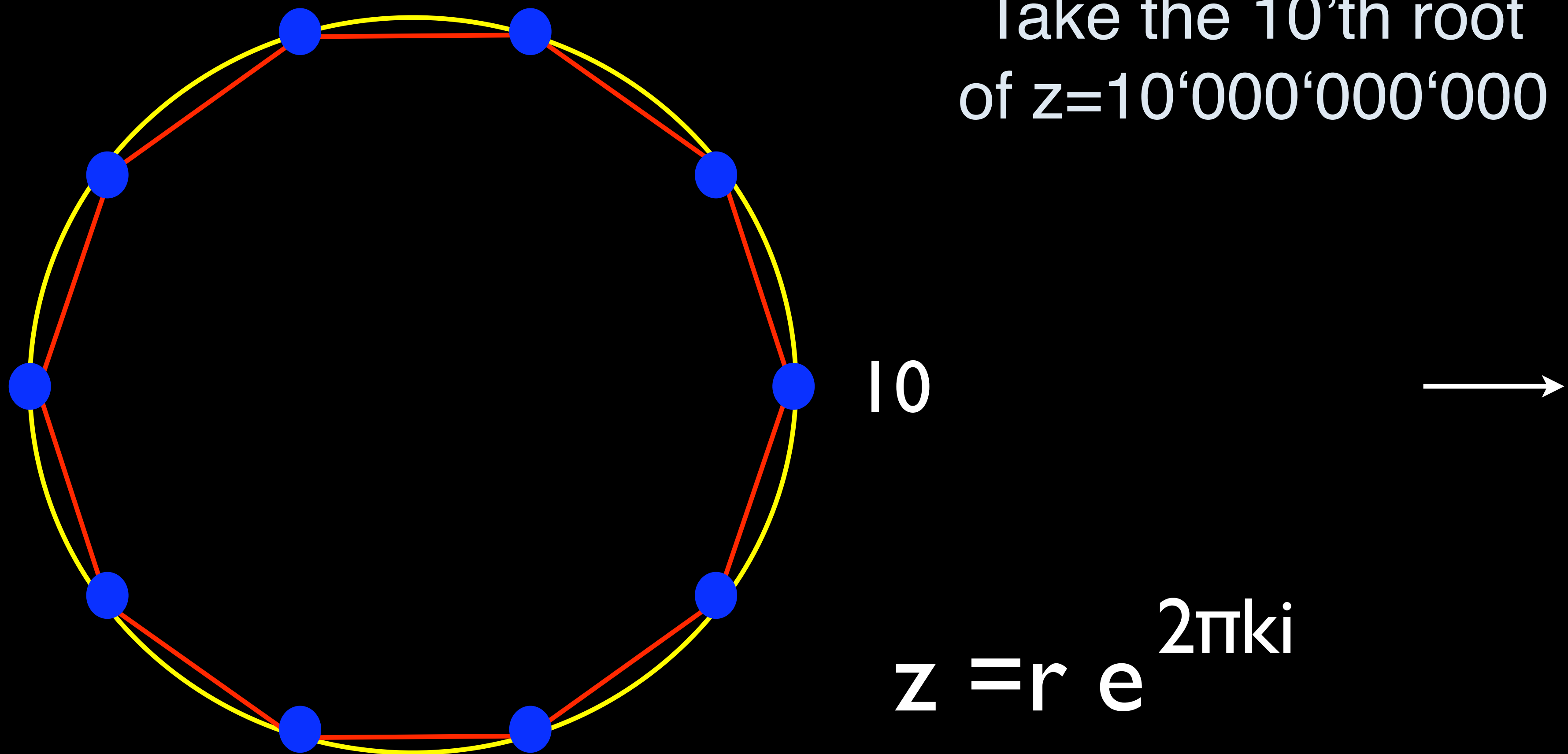


"Take square root of modulus and divide angle by 2".

Every complex number different from 0 has exactly 2 square roots.

Higher roots

Take the 10'th root
of z=10‘000‘000‘000



$$z = r\, e^{2\pi k i}$$

$$\sqrt[n]{z} = r^{1/n}\, e^{2\pi k i/n}$$

Diagonalization

If all eigenvalues of a matrix A are different, then A can be diagonalized.

Example:

$$\begin{matrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 7 & 10 \end{matrix}$$

Is this diagonalizable?

Example:

$$\begin{pmatrix} 11 & 2 & 3 & 4 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Are the following matrices similar?

$$\begin{pmatrix} 3 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 7 & 7 & 5 & 0 \\ 1 & 1 & 7 & 4 \end{pmatrix} \qquad \begin{pmatrix} 4 & 2 & 2 & 2 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

Are the following matrices similar?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The end