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| MWF10 Samik Basu |
| MWF10 Toby Gee |
| MWF11 Oliver knill |
| MWF11 Poning Chen |
| MWF12 Rehana Patel |
| TTH10 Samik Basu |
| TTH1130 Greta Panova |

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Show details of all work except for problems 1-3.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

Problem 1) (20 points) True or False? No justifications are needed.

1) T F Let $A$ and $B$ be two $n \times n$ matrices. Then $A$ and $B$ are similar if and only if they have the same characteristic polynomials.

2) T F Let $A$ be a $n \times n$ matrix. Applying Gauss-Jordan row operations to $A$ does never change the determinant.

3) T F An orthogonal matrix is symmetric $A^T = A$ or anti symmetric $A^T = -A$.

4) T F The eigenvalues of a matrix $A$ do not change under row reduction.

5) T F The eigenvectors of a matrix $A$ do not change under row reduction.

6) T F If $A$ is the matrix of a reflection at a line in space, then $\det(A+I) = 0$ and $\det(A-I) = 0$.

7) T F Every upper triangular matrix can be diagonalized.

8) T F There is a recursion $x_{n+1} = ax_n + bx_{n-1}$ for which $x_n > (1.01)^{2^n}$ for all $n$.

9) T F The sum of two projections is a projection.

10) T F The characteristic polynomial of $A$ is the same as the characteristic polynomial of $A^T$.

11) T F For any matrix, we have $\det(A^5) = \det(A)^5$.

12) T F The trace of a matrix is equal to the product of the eigenvalues.

13) T F There is a projection for which the determinant is equal to 2.

14) T F The matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$.

15) T F If two $2 \times 2$ matrices $A$ and $B$ have the same trace and determinant, then they are similar.

16) T F If $x$ is a solution to the linear equation $Ax = b$, then $x$ is a least square solution to $Ax = b$.

17) T F It is possible that the length of $A^n v$ and the length of $A^{-n} v$ both grow exponentially.

18) T F If an orthogonal matrix $Q$ is symmetric, then $Q$ is diagonal.

19) T F If $A = QR$ is the $QR$ decomposition of a square matrix, then the eigenvalues of $A$ are the diagonal entries of $R$.

20) T F For every invertible $n \times n$ matrix $A$, there is a nonzero $n \times n$ matrix $B$ such that $AB$ is the zero matrix.

Total

| Problem 2) (10 points) |
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No explanations are necessary for this problem.

a) Which matrices are orthogonal, which matrices are symmetric, which matrices are projections? Check everything which applies. It is not excluded that you have to check several properties for each matrix.

| | orthogonal | symmetric | projection | |
|---|---|---|---|---|
| 1) | | | | $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| 2) | | | | $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| 3) | | | | $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 4) | | | | $D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |

b) Which of the following assertions are true?

| true | false | |
|---|---|---|
| | | $A$ is similar to $B$ |
| | | $A$ is similar to $C$ |
| | | $A$ is similar to $D$ |

| | | |
|---|---|---|
| | | $B$ is similar to $C$ |
| | | $B$ is similar to $D$ |
| | | $C$ is similar to $D$ |

c) Match the following matrices with sets of eigenvalues. There is a unique match. It is not necessary to compute the eigenvalues to do so. Enter i),ii) or iii) in the boxes.

| | |
|---|---|
| | $A = \begin{bmatrix} -5 & -9 & -7 \\ 2 & 5 & 2 \\ 4 & 5 & 6 \end{bmatrix}$ |
| | $A = \begin{bmatrix} 5 & -9 & -7 \\ 0 & 5 & 2 \\ 0 & 0 & 6 \end{bmatrix}$ |
| | $A = \begin{bmatrix} 13 & 11 & 13 \\ -2 & -1 & -2 \\ -8 & -7 & -8 \end{bmatrix}$ |

i) $\{3, 2, 1\}$.
ii) $\{1, 0, 3\}$.
iii) $\{6, 5, 5\}$.

Problem 3) (10 points)

Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

a) Find the eigenvalues of $A$ with their algebraic multiplicities.

b) Find the geometric multiplicities of each of the eigenvalues.

c) Find all the eigenvectors.

d) What is $\det(A)$?

Problem 4) (10 points)

Find the function $y = f(x) = a\cos(\pi x) + b\sin(\pi x)$, which best fits the data

| x | y |
|---|---|
| 0 | 1 |
| 1/2 | 3 |
| 1 | 7 |

Problem 5) (10 points)

a) Find all the eigenvalues $\lambda_1, \lambda_2$ and $\lambda_3$ of the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) Find a formula for $\mathrm{tr}(A^n)$.

Problem 6) (10 points)

a) (5 points) Find an eigenbasis of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

b) (5 points) Do Gram-Schmidt orthogonalization on the basis $\mathcal{B} = \{v_1, v_2, v_3\}$ you just got. Write down the $QR$ decomposition of the matrix $S$ which contains the basis $\mathcal{B}$ as column vectors.

| Problem 7) (10 points) |
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a) (5 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 4 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

b) (5 points) Find the determinant of the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 & 1 & 4 \end{bmatrix}$$

| Problem 8) (10 points) |
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We want to find a formula for the general term $x_n$ in the recursion

$$x_{n+1} = x_n + 3x_{n-1}/4$$

if $x_0 = 0, x_1 = 1$. This is the case of a Fibonacci recursion, in which only 3/4 of the previous generation has kids.

a) Write the recursion in the form $v_{n+1} = Av_n$ for vectors $v_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in the plane.

b) Find the eigenvalues $\lambda_+, \lambda_-$ and eigenvectors $v_+, v_-$ of $A$.

c) Write $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $v_0 = av_+ + bv_-$.

d) Find $v_n = A^n v_0$ and so $x_n$.

Problem 9) (10 points)

Our goal is to find the determinant of

$$A = \begin{bmatrix} 101 & 1 & 1 & 1 & 1 & 1 \\ 2 & 102 & 2 & 2 & 2 & 2 \\ 3 & 3 & 103 & 3 & 3 & 3 \\ 4 & 4 & 4 & 104 & 4 & 4 \\ 5 & 5 & 5 & 5 & 105 & 5 \\ 6 & 6 & 6 & 6 & 6 & 106 \end{bmatrix}$$

a) The matrix $A - 100I_6$ has an eigenvalue 0. Find its algebraic multiplicity.

b) The matrix $A^T$ has an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find the corresponding eigenvalue $\lambda$.

c) Why does $A$ also have the same eigenvalue $\lambda$?

d) You have found all the eigenvalues of $A - 100I_6$. What are the eigenvalues of $A$?

e) Find the determinant of $A$.