Name:

MWF10 Samik Basu
MWF10 Toby Gee
MWF11 Oliver knill
MWF11 Poning Chen
MWF12 Rehana Patel
TTH10 Samik Basu
TTH1130 Greta Panova

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications needed

1) T F If A is invertible, then A^3 is invertible.

2) T F If A^2 is invertible, then A is invertible.

3) T F There is a matrix A and a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution and $\ker(A) = \{\vec{0}\}$.

4) T F If A is a 3×3 matrix which is a rotation around a line in $\mathbb{R}^3$ then the columns of A form a basis of $\mathbb{R}^3$.

5) T F The rank of a diagonal matrix A equals the number of non-zero entries in A.

6) T F Row reduction produces a diagonal matrix which has leading 1 at the places where the original entries were not zero.

7) T F $A = \begin{bmatrix} 4/13 & 6/13 \\ 7/13 & 9/13 \end{bmatrix}$ is a projection onto the line spanned by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

8) T F The composition of a shear and a rotation in the plane is invertible.

9) T F The linear space of cubic polynomials $ax^3 + bx^2 + cx + d$ has dimension 3.

10) T F If A is a 5×5 matrix such that $A^2 = I_5$, (where I_5 is the identity matrix), then A is invertible.

11) T F If A is a 2×2 matrix such that $A^3 = I_2$, then A is the identity matrix I_n.

12) T F There exists an invertible 3×3 matrix A such that 7 of its entries are 0.

13) T F It can happen that in $\mathrm{rref}(A)$ there are rows for which all entries are nonzero.

14) T F It is possible that $\mathrm{rref}(A)$ has columns for which all entries are nonzero.

15) T F The rank of a 3×4 matrix may be 4.

16) T F The rank of a 4×3 matrix may be 4.

17) T F There is an invertible 2×2 matrix S such that $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

18) T F The number of leading 1 entries in $\mathrm{rref}(A)$ is the dimension of the image of A.

19) T F If A and B are invertible $n \times n$ matrices, then so is $A - B$.

20) T F There exists a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ for which $\ker(T) = \mathrm{im}(T)$.

Total

Problem 2) (10 points)

Which of the following matrices are in reduced row-echelon form? We do not need explanations in this question.

	Matrix	IS	IS NOT		Matrix	IS	IS NOT
a)	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$			b)	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
c)	$\begin{bmatrix} 1 & 6 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$			d)	$\begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
e)	$\begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$			f)	$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$		
g)	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$			h)	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
i)	$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$			j)	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$		

Problem 3) (10 points)

In problems a)-b), you have to find all the solutions using Gauss-Jordan elimination.

a) (4 points) Find all solutions of $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 5 \\ 2 & 3 & 4 \end{bmatrix} .$$

b) (4 points) Find all solutions to the following system of linear equations

$$\begin{aligned} x + y + z + w &= 4 \\ x - y + z - w &= 2 \end{aligned}$$

$$2x + 2y + 2z + 2w = 8$$

c) (2 points) You have a solution $x \in \mathbb{R}^9$ of a system of linear equations $Ax = b$, where A is a 7×9 matrix, and b is a given vector in $\mathbb{R}^7$. Is it possible to find an other solution $Ay = b$, where y is different from x?

Problem 4) (10 points)

Which of the following sets are linear subspaces of some $\mathbb{R}^n$? You do have to give explanations.

1) (2 points) The image of the transformation $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

2) (2 points) The kernel of the projection from $\mathbb{R}^3$ onto the xy-plane.

3) (2 points) The solutions of the equation $2x + 3y - 5z = 12$ in $\mathbb{R}^3$.

4) (2 points) All the vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ which satisfy $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$.

5) (2 points) All the points $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{x}\}$, where A is a given $n \times n$ matrix.

Problem 5) (10 points)

Let A be a shear in the plane along the x-axis which maps $\vec{e}_1$ to $\vec{e}_1$ and sends $\vec{e}_2$ to $2\vec{e}_1 + \vec{e}_2$. One calls this transformation also a horizontal shear. Let B the the projection onto the x-axis.

a) (4 points) Find the matrices A, B, AB and BA.

b) (3 points) What is A^{100}?

c) (3 points) What is $(AB)^{10}$?

Problem 6) (10 points)

a) (3 points) Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is a reflection over the x-axis. Find the matrix of T.

b) (4 points) Let S be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ obtained by first reflecting over the x-axis, then reflecting over the y-axis, and finally reflecting over the line $y = x$. Find the matrix A of S.

c) (3 points) Find the inverse of A, where A is the matrix you found in part (b).

Problem 7) (10 points)

a) (3 points) Find a basis for the kernel of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

b) (3 points) Find a basis for the image of the following 4×6 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}.$$

What is the dimension of the image and the dimension of the kernel?

c) (4 points) Find the dimension of the image and kernel of the following 4×100 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \cdots & 99 & 100 \\ 2 & 3 & 4 \cdots & 100 & 101 \\ 3 & 4 & 5 \cdots & 101 & 102 \\ 4 & 5 & 6 \cdots & 102 & 103 \end{bmatrix}.$$

Problem 8) (10 points)

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $\mathcal{B}$ be the basis $\{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$.

a) (3 points) Find the coordinates of $\vec{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ in the basis $\mathcal{B}$.
In other words, find the $\mathcal{B}$-coordinates of $\vec{v}$.

b) (4 points) The matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ defines a linear transformation $T(\vec{x}) = A\vec{x}$. What is the $\mathcal{B}$-matrix of T?

c) (3 points) Is there a different basis $\mathcal{B}$ such that the $\mathcal{B}$-matrix of T is $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$? Explain briefly. (Here, T is the linear transformation defined in part (b).)

Problem 9) (10 points)

You are given a matrix A which is a 5×6 matrix of rank 5. In each of the following questions, we need not only the answer but also a short explanation.

a) (2 points) Can the matrix A be invertible ?

b) (3 points) What is the dimension of the image?

c) (3 points) What is the dimension of the kernel?

d) (2 points) How many solutions will the equation $A\vec{x} = \vec{b}$ have?

Problem 10) (10 points)

A general shear is a linear transformation T for which there is a vector $\vec{v}$ with $T(v) = v$ and such that for all vectors $T(x) - x$ is parallel to v. Is the linear transformation $T(x) = A(x)$ with

$$A = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

a general shear? If so, find the line along which it is centered. It is enough to give a nonzero vector $\vec{v}$ in that line.