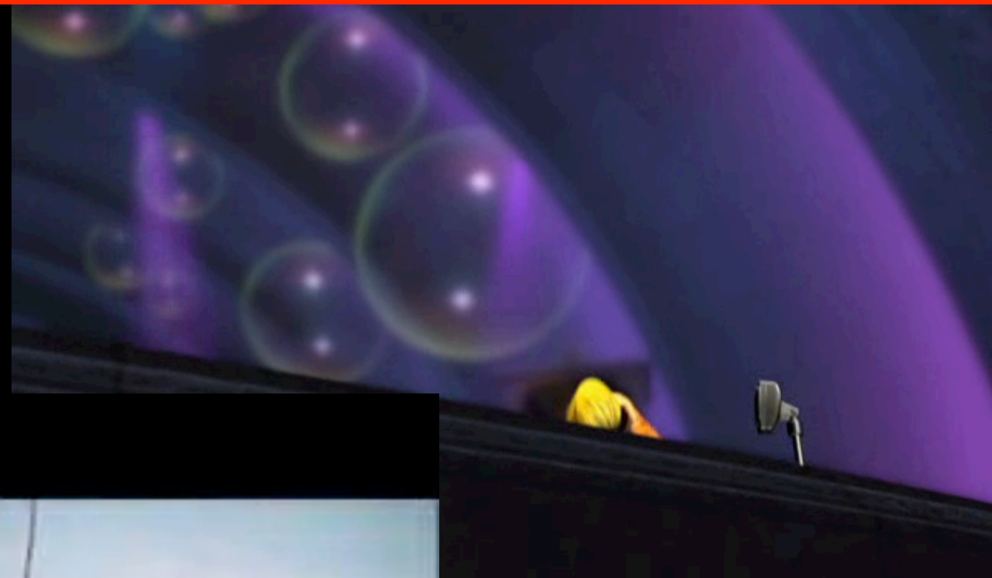
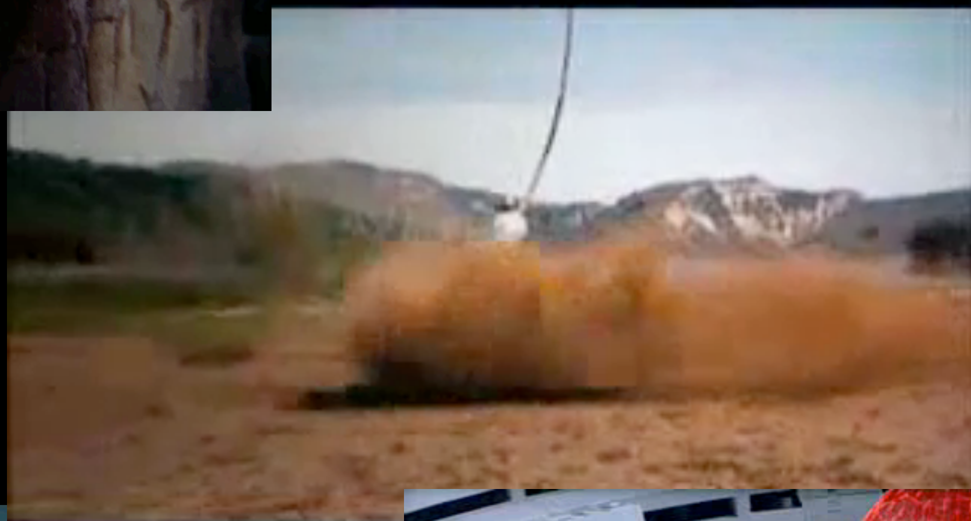
Math21b

Review to first midterm

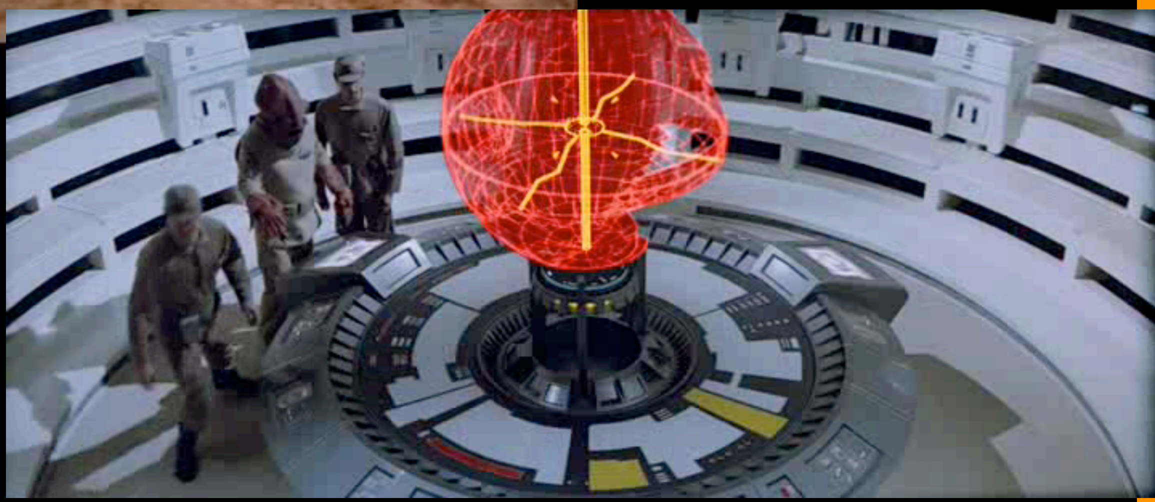
Spring 2009

Oliver Knill, March 1, 2009





What theme?



Why not ....

the number

8

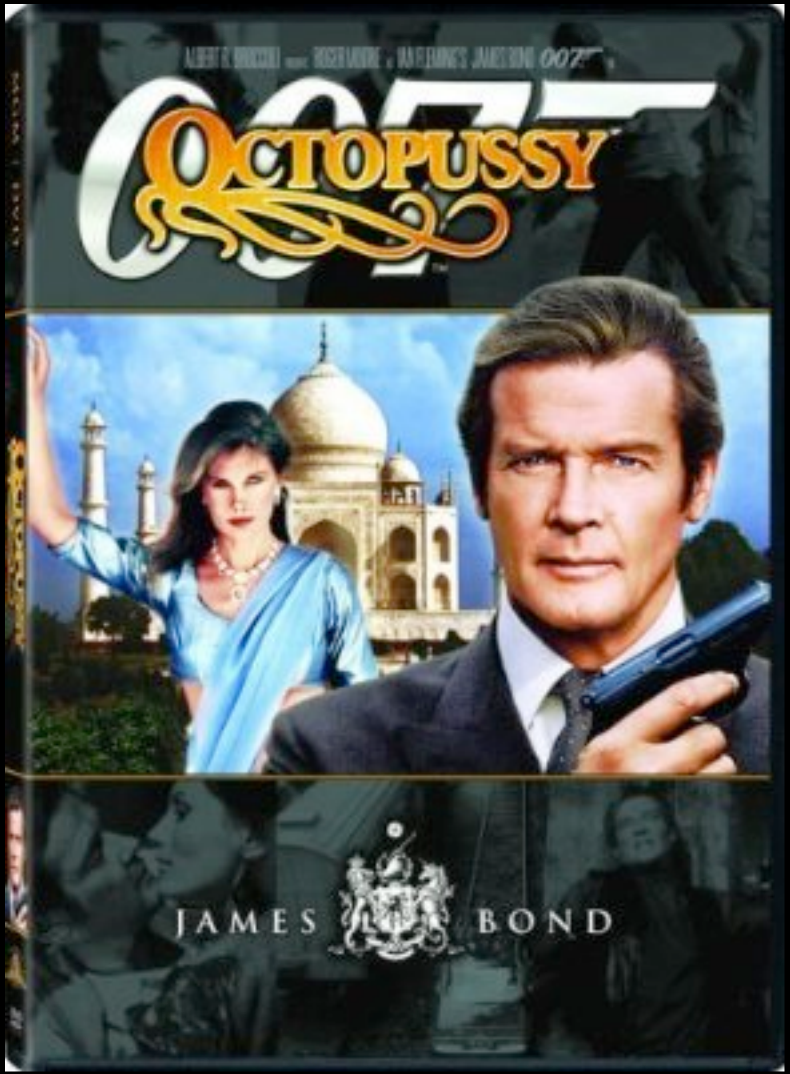
It has been a good time for the number 8

1950

pieces of eight

1950: cinderella and 7 dwarfs

1983: octopussy



2002: 8 mile

2008: the octobox

2009: naughty octopus

2009: the octomom


Pieces of eight
octomom
8 Mile
BREAKING NEWS
LEWIS: IT'S GOING TO BECOME MORE DIFFICULT FOR MIDSIZED BANKS TO COMPETE
octobox
CNN
KCAL
Videos Like This
naughty octopus

I. Matrices

column picture

$$Ax = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 v_1 + x_2 v_2 + \ldots + x_m v_m$$


AIG 5.90 ▼ 6.24
WM 2.12 ▼ 0.60
UBS 18.02 ▼ 2.66
INDU -252.08
INDP 11168.94
BREAKING NEWS
LEWIS: IT'S GOING TO BECOME MORE DIFFICULT FOR MIDSIZED BANKS TO COMPETE
General Electric [GE] 14.3K@25.72 ▼ 1.03
[SDS] 10K@70.80 ▲ 3.39
Ultra Financials [UYG]
CNBC
96.47
CRUDE OIL

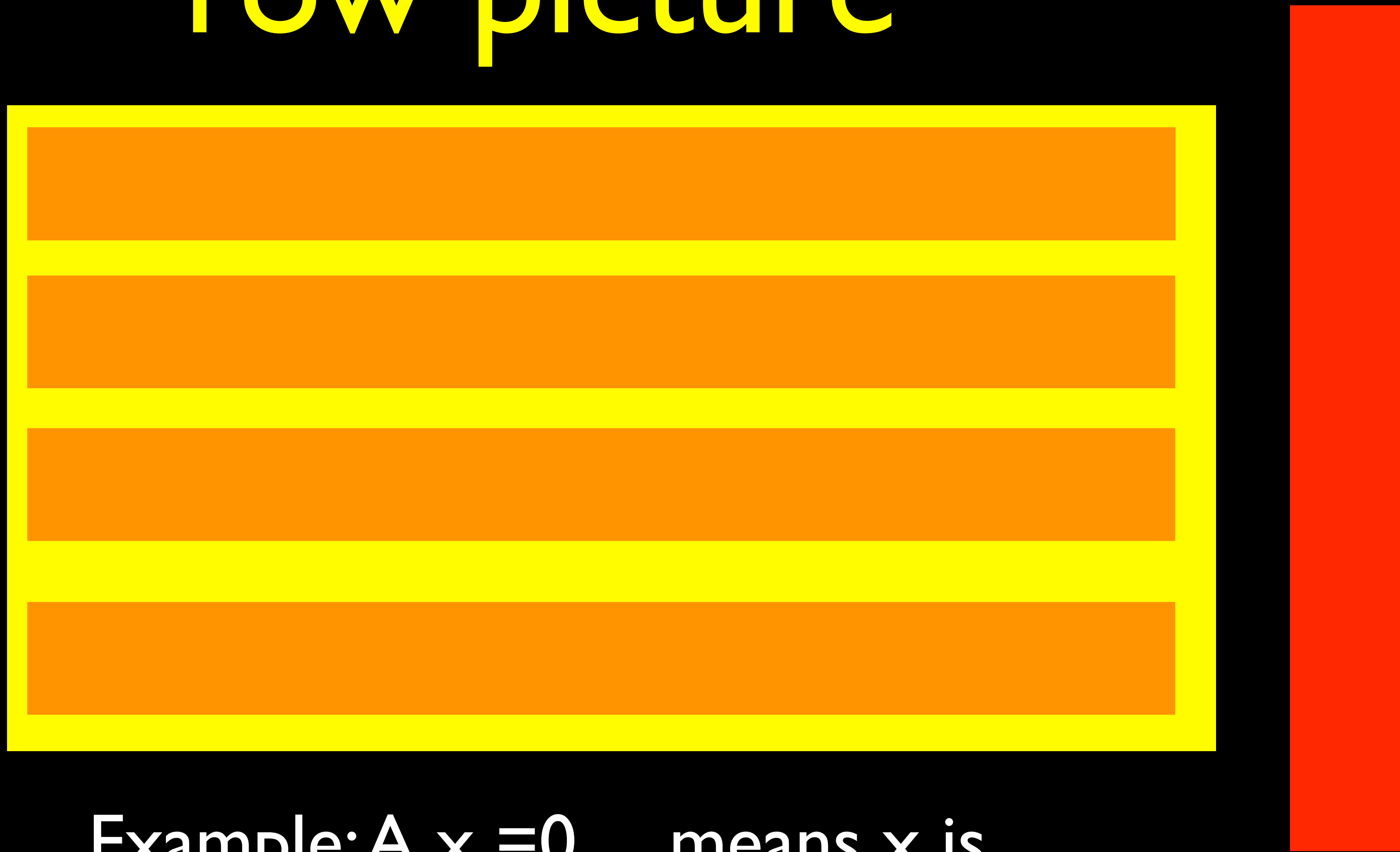
Example: the Octobox

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 6 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

row picture

$Ax=$



Example: $A x = 0$, means x is perpendicular to row column vectors .

Example: $A x = b$, means b_k is the dot product of the k'th row with x.

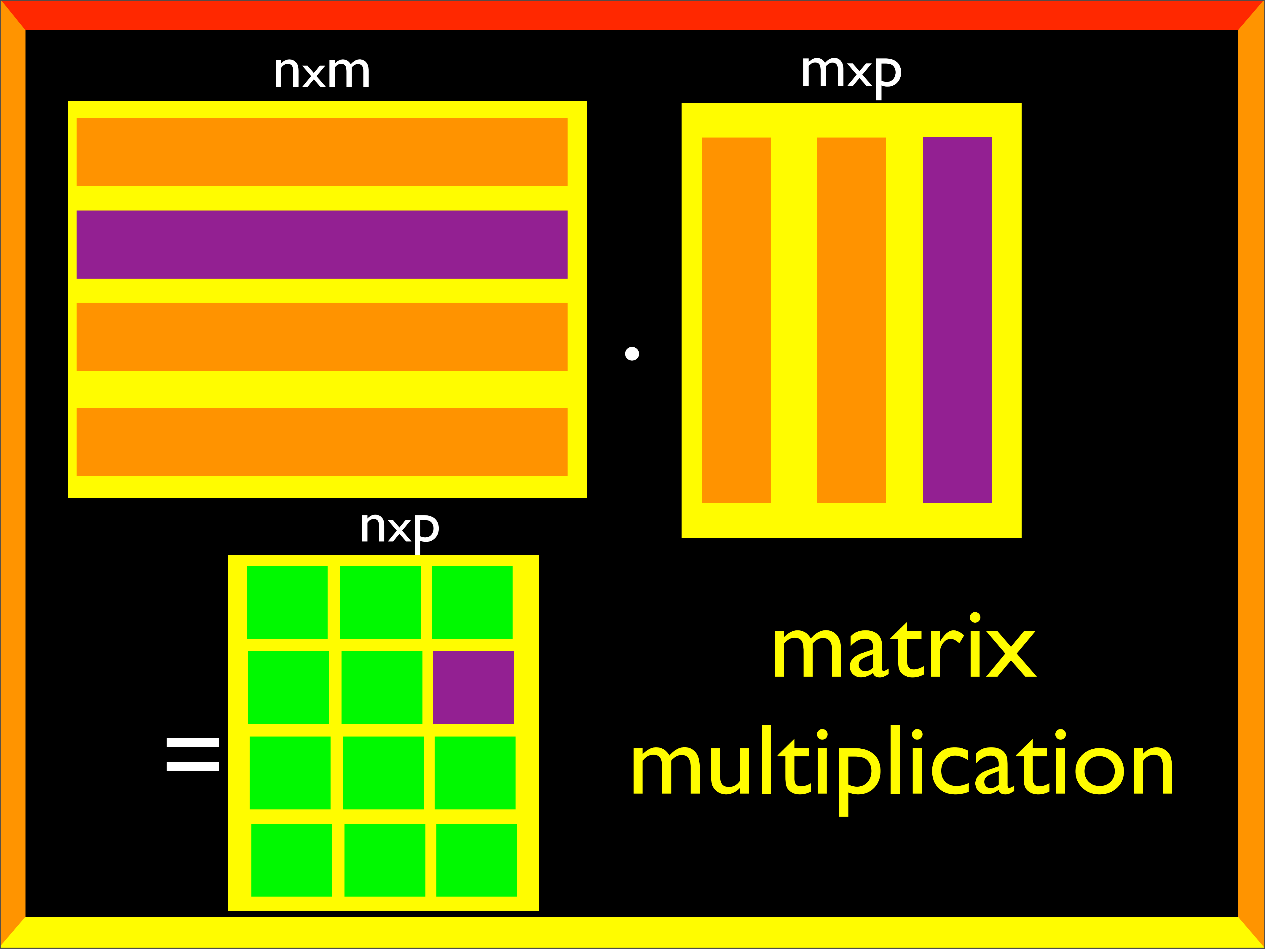
Example

to find all vectors perpendicular to these two octobox vectors:

1	1
2	4
4	6
6	8

1	2	4	6
1	4	6	8

solve A x = 0, where A contains the vectors as rows.


nxm
mxp
nxp
=
matrix
multiplication

1x8 · 8x1 = 1x1

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 36 \end{bmatrix}$$

8x1

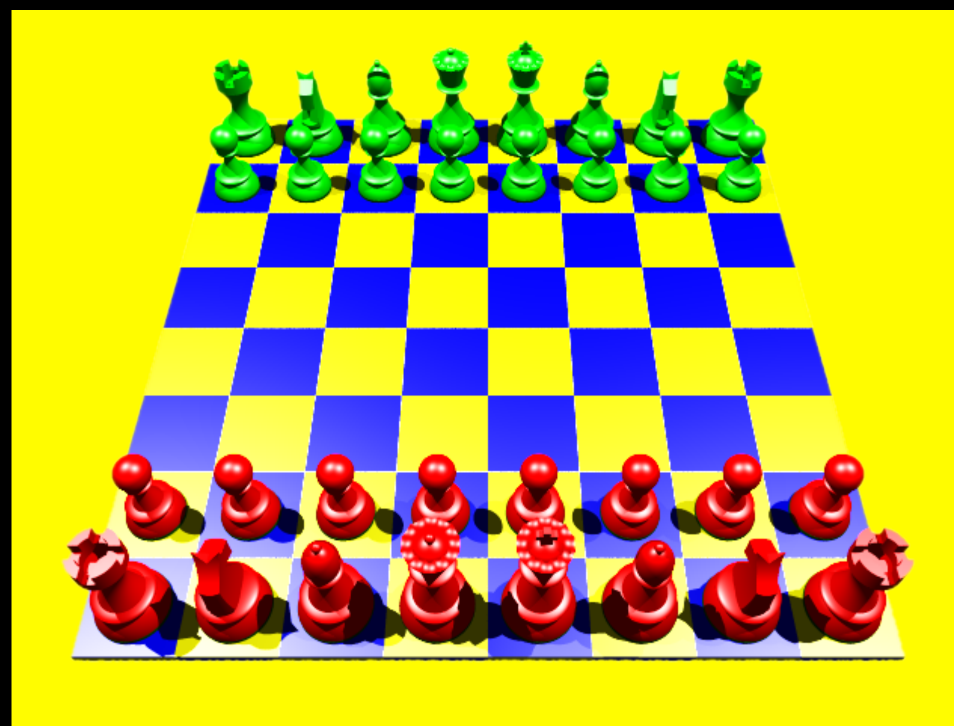
·

1x8

1 2 3 4 5 6 7 8

=

8x8



Problem:

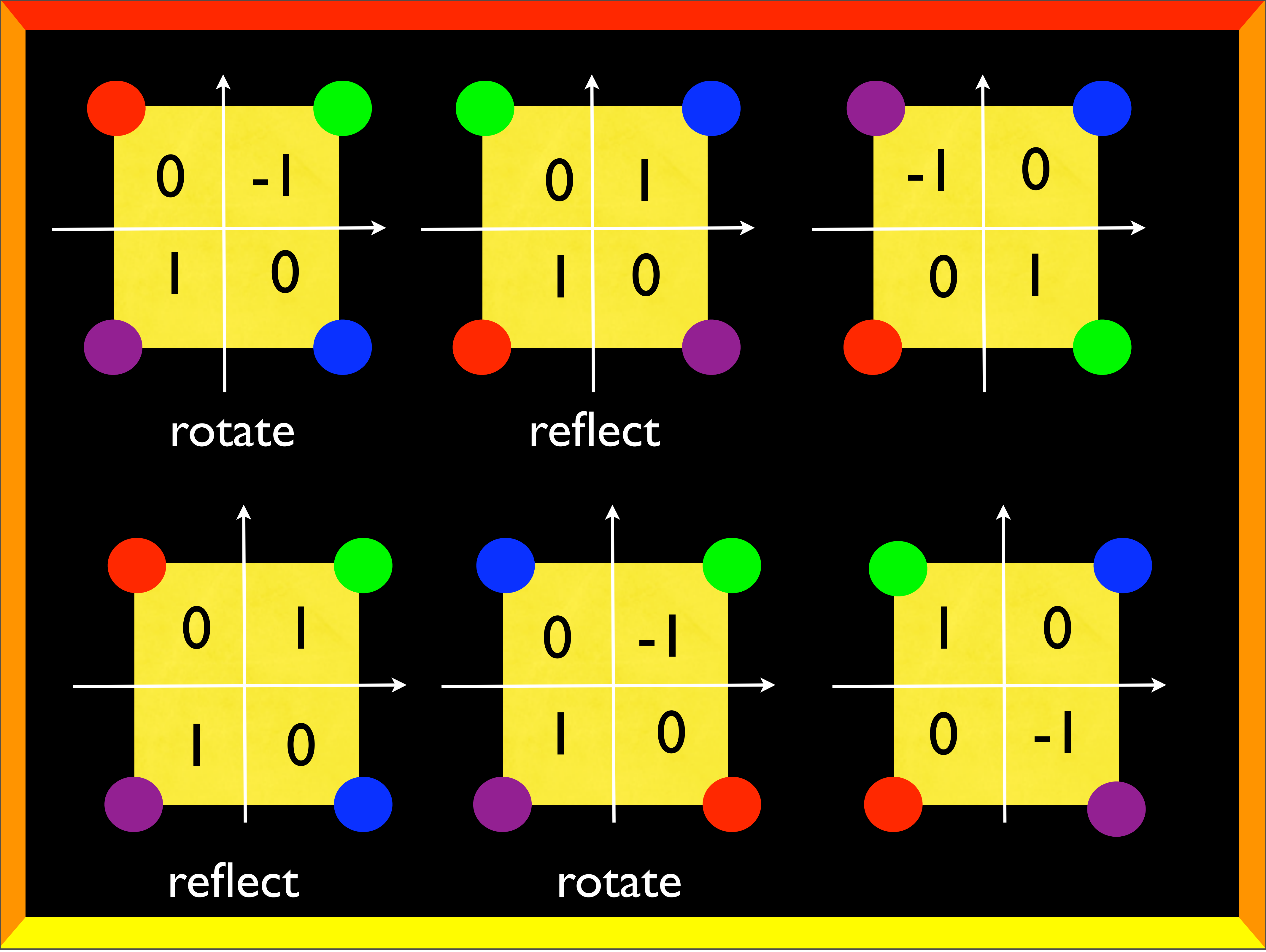
A 3 x 8 matrix

B 8 x 3 matrix

Is A B defined?

Is BA defined?

Is A A defined?



Monday, March 2, 2009

Matrix algebra:

With nxn matrices A,B,C,D,... one can work as with numbers

$$A+B = B+A$$
$$a\,(A+B) = aA + aB$$
$$A(B+C) = AB + AC$$
$$A\,(B\,C) = (A\,B)\,C$$

etc

except for two things in general we have:

$$A\,B \neq B\,A$$

and

A^{-1} might not exist even for nonzero A

if A,B are invertible: $(A\,B)^{-1} = B^{-1}A^{-1}$

True or False?

A,B, arbitrary nxn matrices

$$(A + B)(A - B) = A^2 - B^2$$

$$(I + A + A^2 + A^3) = (I - A^4)(I - A)^{-1}$$

assuming $(I - A)$ is invertible

$$I = I_n$$

Linear Equations

Systems of linear equations

$$A\,x = b$$

nxn matrix A

exactly one solution

$$x = A^{-1}b$$

Matrix equation

$$A\,X = B$$

$$X = A^{-1}B$$

blackboard problem

Find all the solutions of

$$x+y+u+v = 8$$
$$x-y+u-v = 8$$

Find all the solutions of

$$x+y+u+v = 8$$
$$x - y + u - v = 8$$

equations

x	y	u	v	
1	1	1	1	8
1	-1	1	-1	8

augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 8 \\ 1 & -1 & 1 & -1 & 8 \end{bmatrix} = [A \mid b]$$

columns: x, y, u, v

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 8 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \mathrm{rref}([A \mid b])$$

columns: x, y, u (r), v (s)

possibilites $A x = b$



one solution
consistent

zero solutions
inconsistent

infinitely many
consistent

rank(A)=n

rank(B)>rank(A)

B=[A|b]

augmented matrix

How many solutions? augmented matrix is

1	4	1	6	1	4	1	2
0	0	1	2	0	0	1	3
0	0	0	1	0	0	0	1
0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	1
0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0

Row reduction

Gauss-Jordan elimination

Scale a row

Swap two rows

Subtract row from other row

S S S

blackboard problem

row reduce:

$$\begin{bmatrix} 8 & 8 & 8 \\ 8 & 0 & 8 \\ 8 & 8 & 8 \\ 8 & 0 & 8 \\ 8 & 8 & 8 \end{bmatrix}$$

Row reduce the 8 matrix

8	8	8
8	0	8
8	8	8
8	0	8
8	8	8

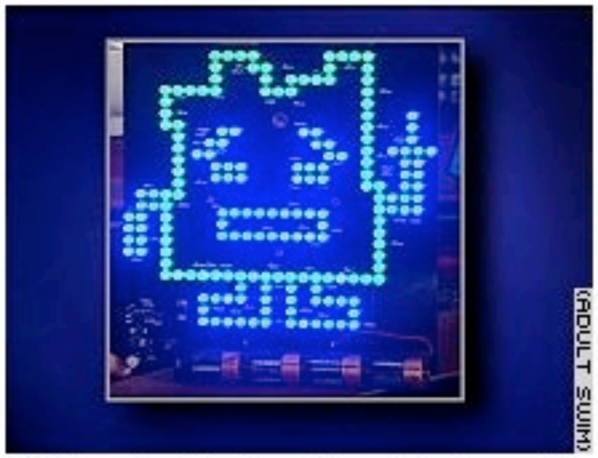
The Boston milkshake scare:

U.S.

Tools: Save | Print | E-mail | Most Popular

Ad campaign triggers bomb scare in Boston

POSTED: 6:31 p.m. EST, January 31, 2007



Lightboards featuring a Mooninite character have been in place for weeks in 10 U.S. cities, a Turner Broadcasting statement says.

STORY HIGHLIGHTS

- Packages were promotional material for Adult Swim network show
- Devices included a light board that displayed a character on the show
- The devices were placed in 10 cities across the country

Adjust font size: − +

BOSTON, Massachusetts (CNN) -- Electronic light boards featuring an adult-cartoon character triggered bomb scares around Boston on Wednesday, spurring authorities to close two bridges and a stretch of the Charles River before determining the devices were harmless.

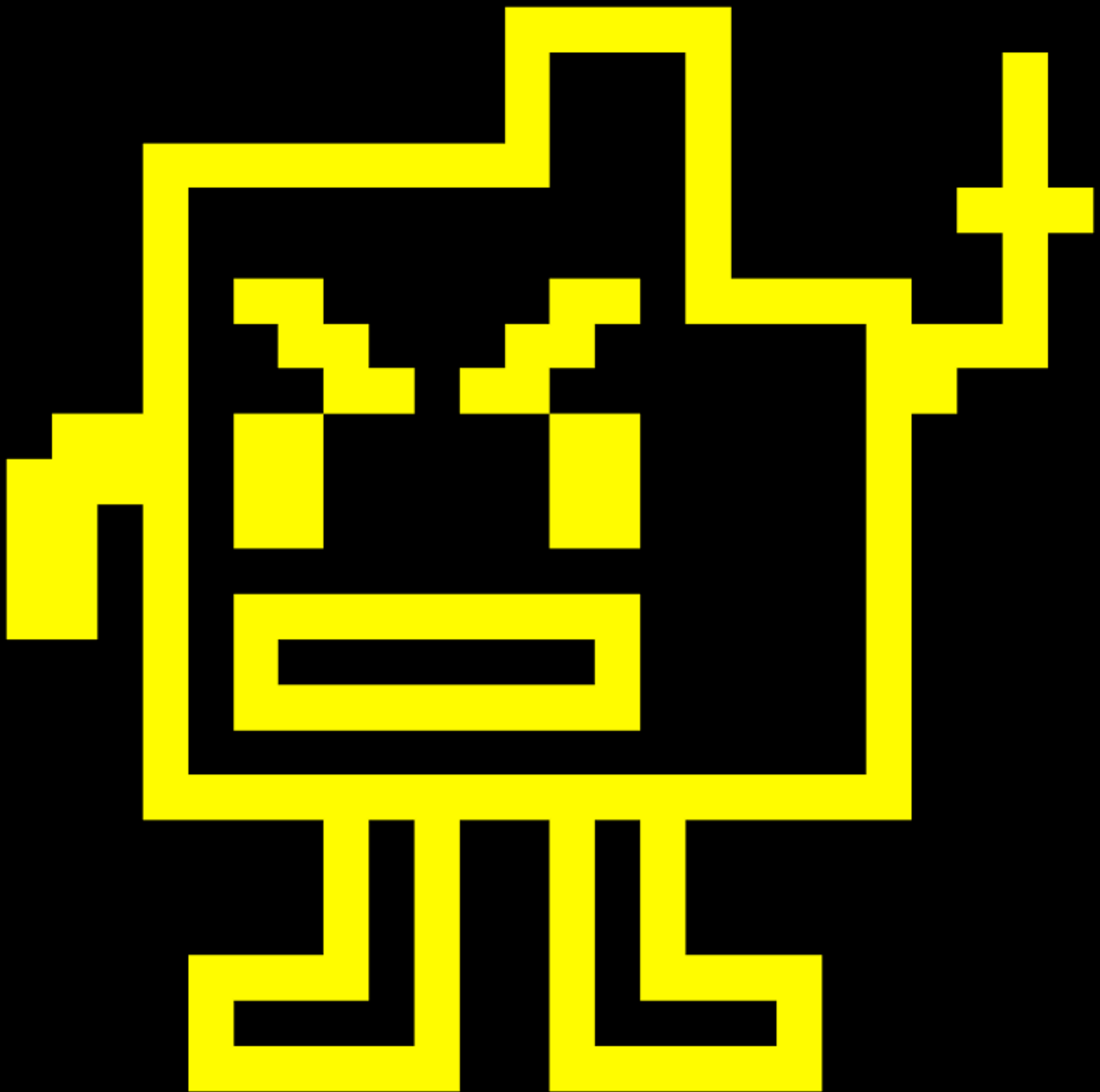
Turner Broadcasting Co., the parent company of CNN, said the devices contained harmless magnetic lights aimed at promoting the Adult Swim network's late-night cartoon "Aqua Teen Hunger Force." Law enforcement sources said the devices displayed one of the Mooninites, outer-space delinquents who appear frequently on the show, greeting visitors with a raised middle finger.

"While the concern is lessened as a result of the investigation, I'd like to remind

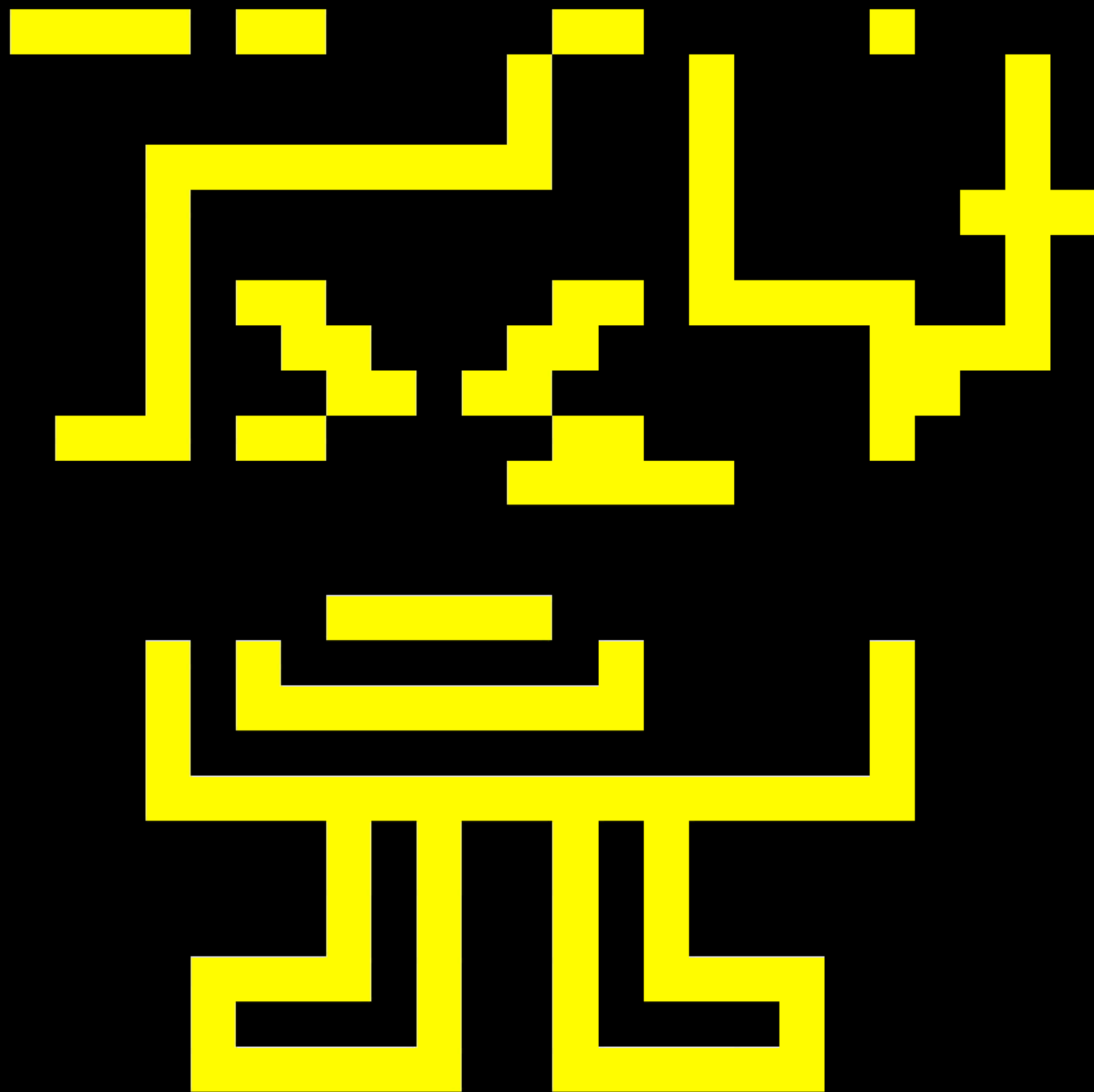

BRIDGE
STATE

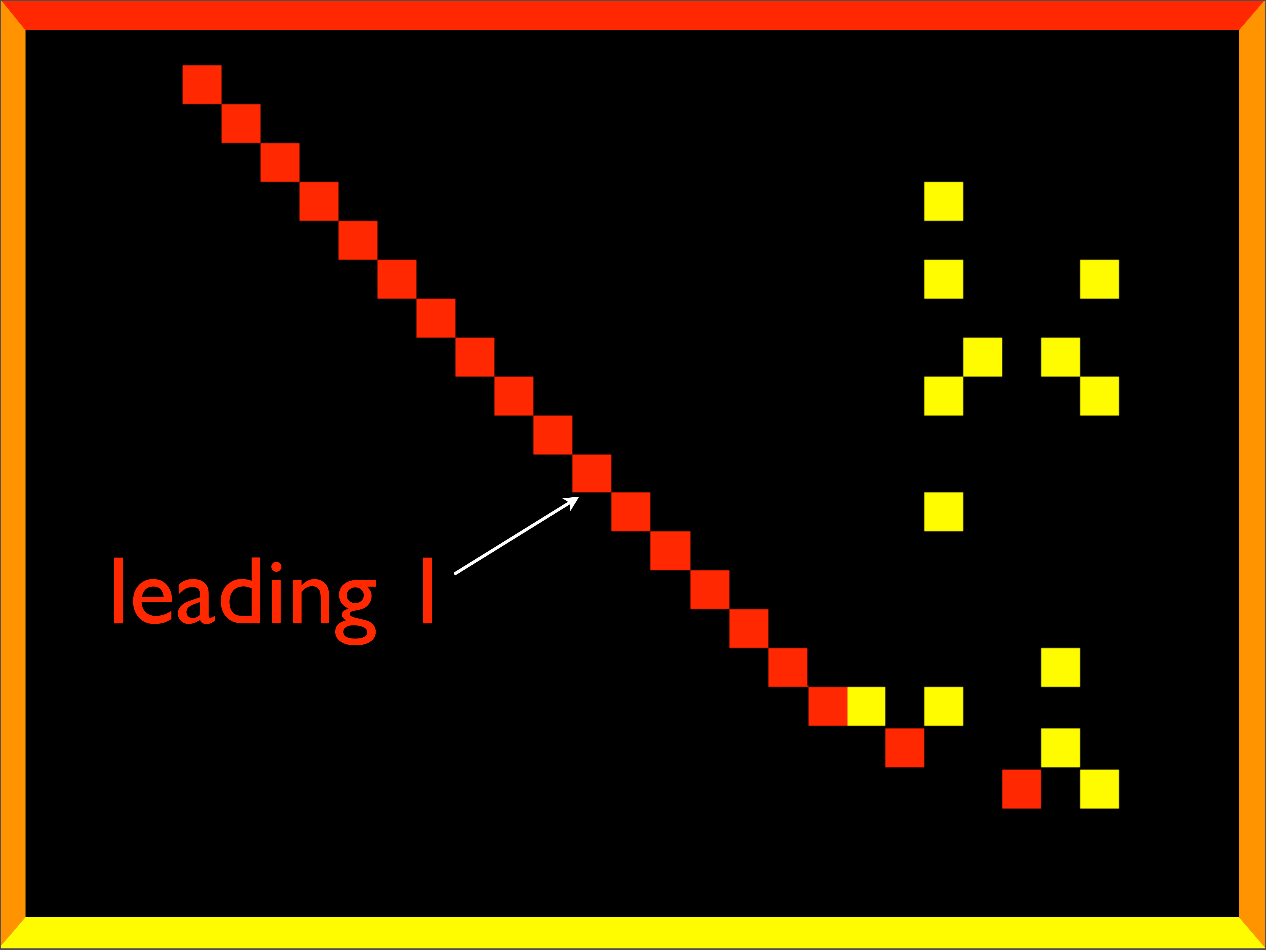
The milkshake as a matrix

Lets row reduce it!

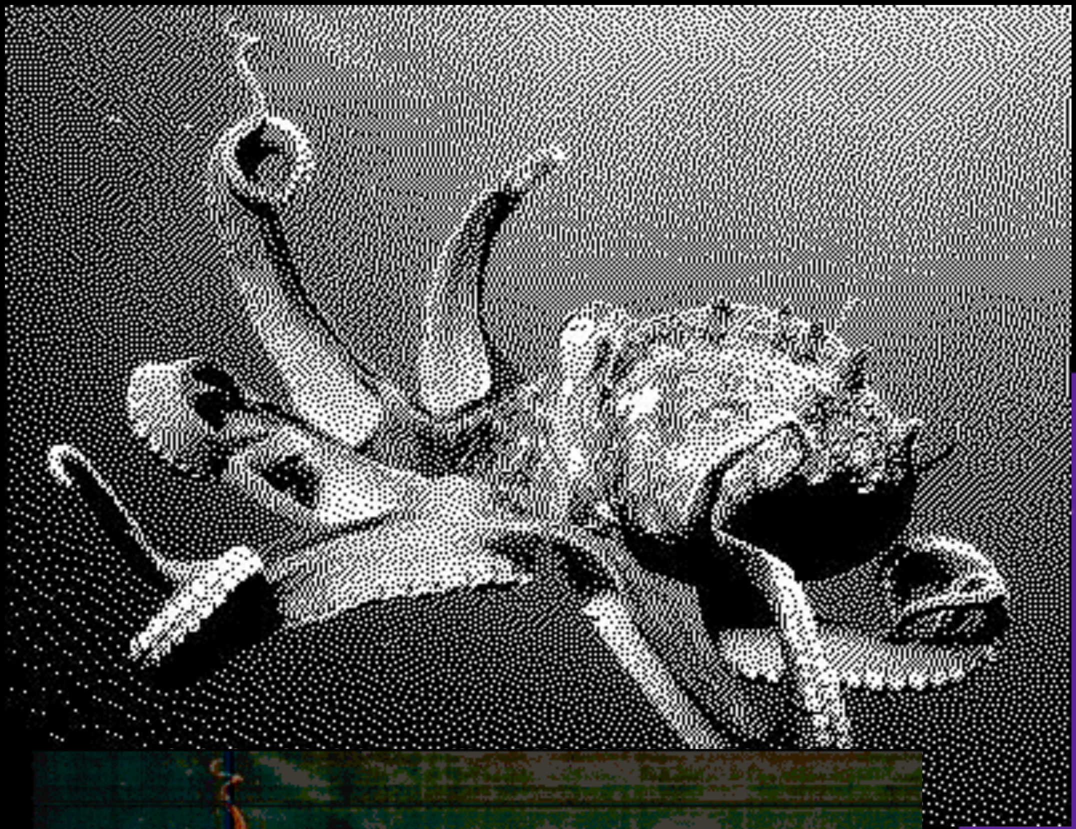


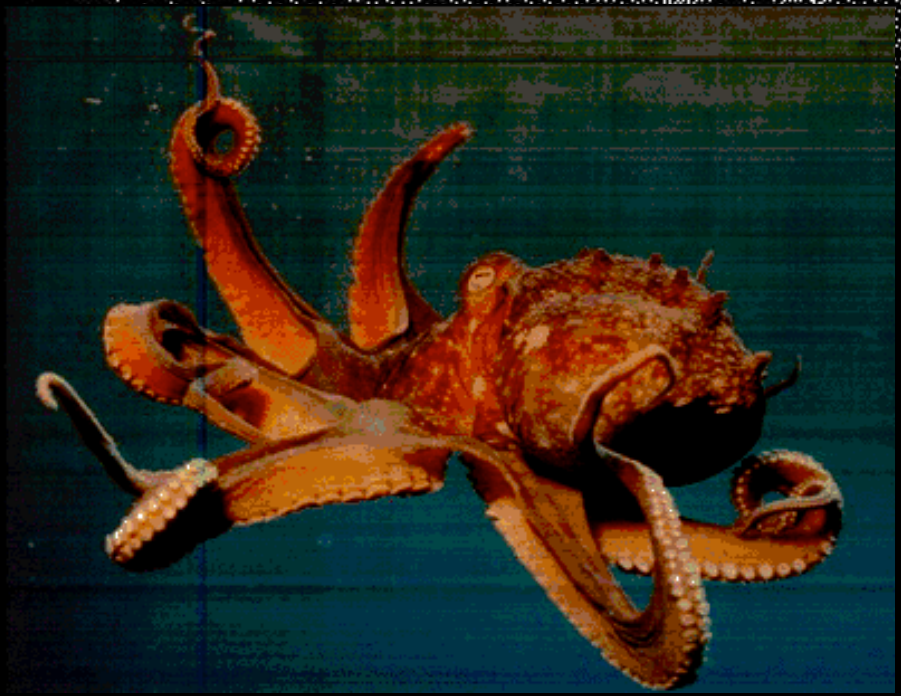
The rref death of a milkshake

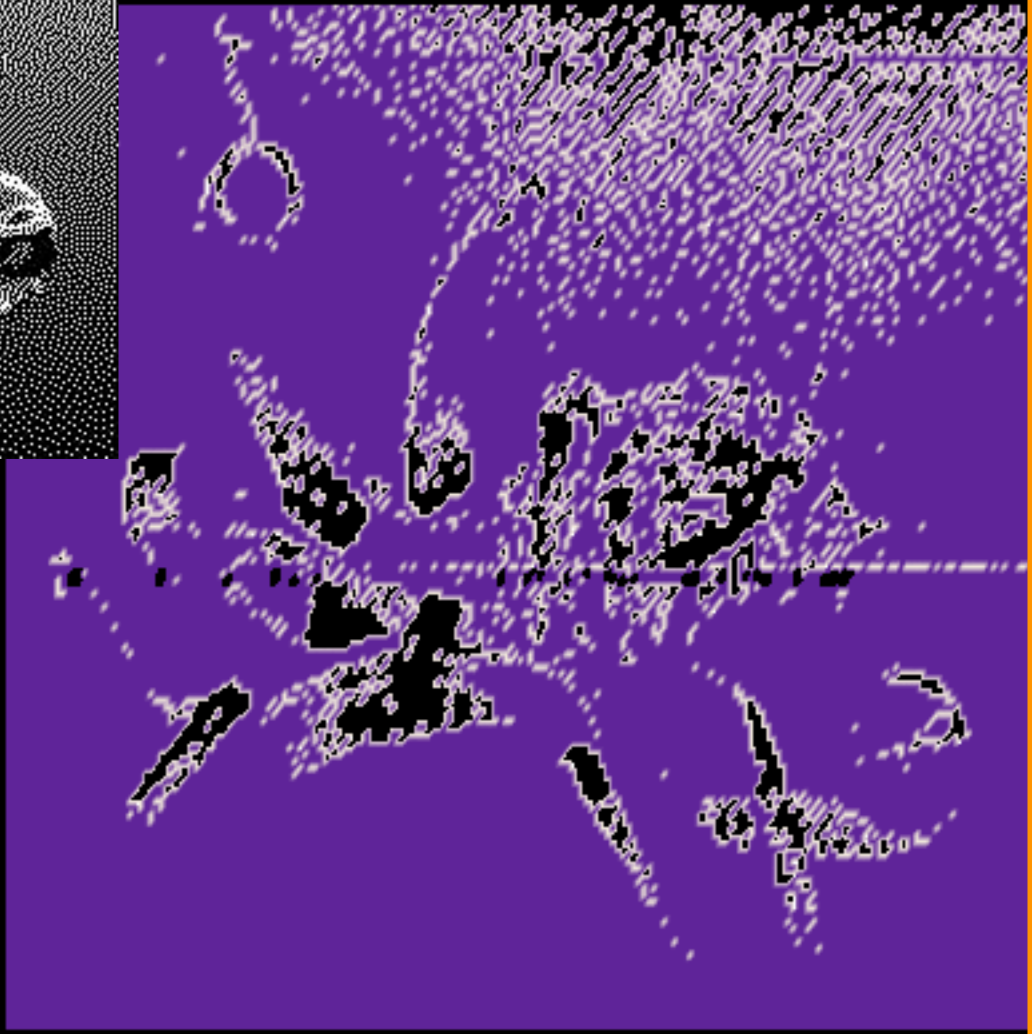



leading 1

works also on octopi







Row reduced echelon form

1. Every first nonzero element in a row is 1
2. Leading columns otherwise only contain 0's
3. Every row above leading row leads to the left

"Leaders like to be first, do not like other leaders in the same column and like leaders above them to be to their left."

Row reduced echelon form?

1	8	8	0	0
0	0	0	0	0
0	0	0	1	0
0	0	0	0	1

Row reduced echelon form?

1	0	8	8	8
0	1	8	8	8
0	0	0	0	0
0	0	0	0	0

Row reduced echelon form?

1	8	0	0	0	8
0	0	1	0	0	8
0	0	0	1	0	8
0	0	0	0	1	8
0	0	0	0	0	0

Inverting a matrix

blackboard problem

Find $\begin{bmatrix} 8 & 0 & 8 \\ 8 & 8 & 0 \\ 8 & 8 & 8 \end{bmatrix}^{-1}$

invert the following matrix

8	0	8
8	8	0
8	8	8

“7 pieces of 8”

$$\begin{pmatrix} 8 & 0 & 8 \\ 8 & 8 & 0 \\ 8 & 8 & 8 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1/8 & 1/8 & -1/8 \\ -1/8 & 0 & 1/8 \\ 0 & -1/8 & 1/8 \end{pmatrix}$$

Which 2x2 matrices are their own inverse?

4 examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

are there more?

There are more!

change basis

if $A = A^{-1}$

then $A^2 = I$

if $B = S^{-1} A\, S$ then

$$B^2 = S^{-1} A\, S\, S^{-1} A\, S = S^{-1} A^2 S$$

$$= S^{-1} I\, S = I$$

Matrix algebra

$$A X = B \rightarrow X = A^{-1}B$$

$$(A X)^{-1} + B = C \rightarrow X = A^{-1}(C-B)^{-1}$$

Multiply from the left or the right

II. Linear transformations

T plays well with 0, addition and scalar multiplication:

$$Tx = Ax$$

$$T(0)=0$$

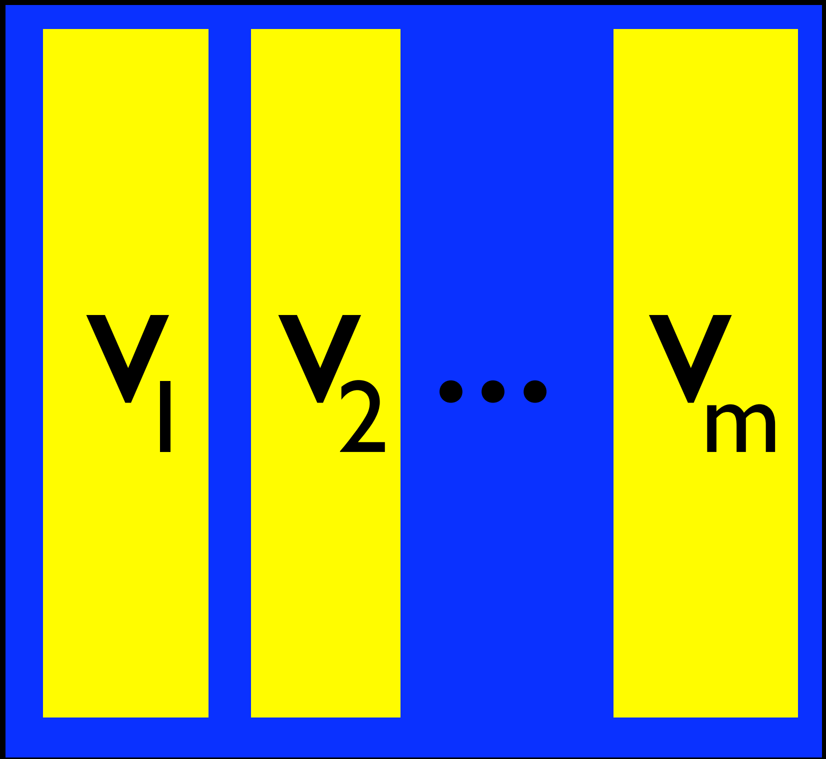
$$T(x+y) = T(x) + T(y)$$

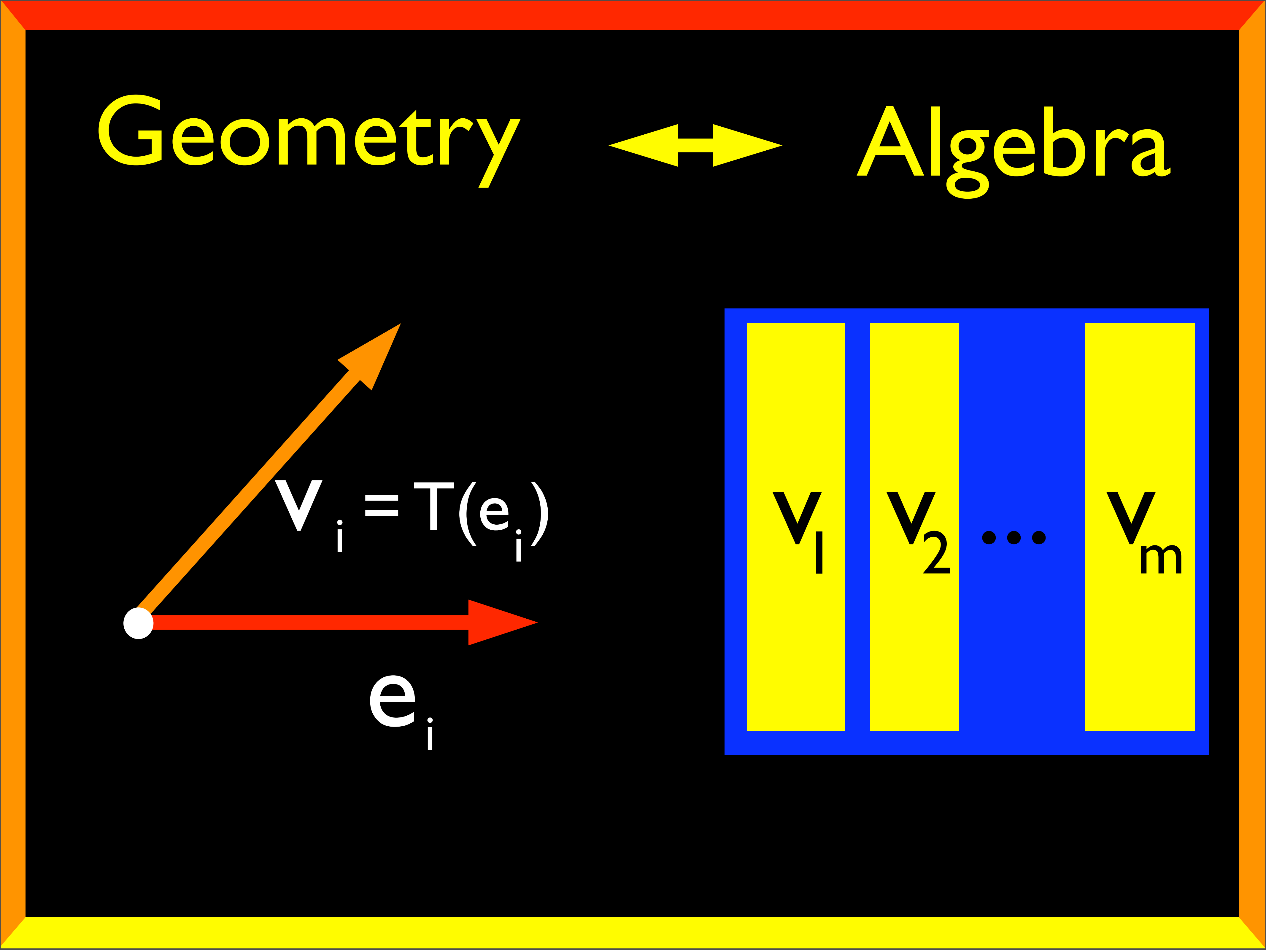
$$T(\lambda x) = \lambda T(x)$$

How do we express T as a matrix?

Key Fact:

The columns of A are the images of the basis vectors.




Geometry ⟷ Algebra
$v_i = T(e_i)$
e_i
v_1 v_2 ⋯ v_m

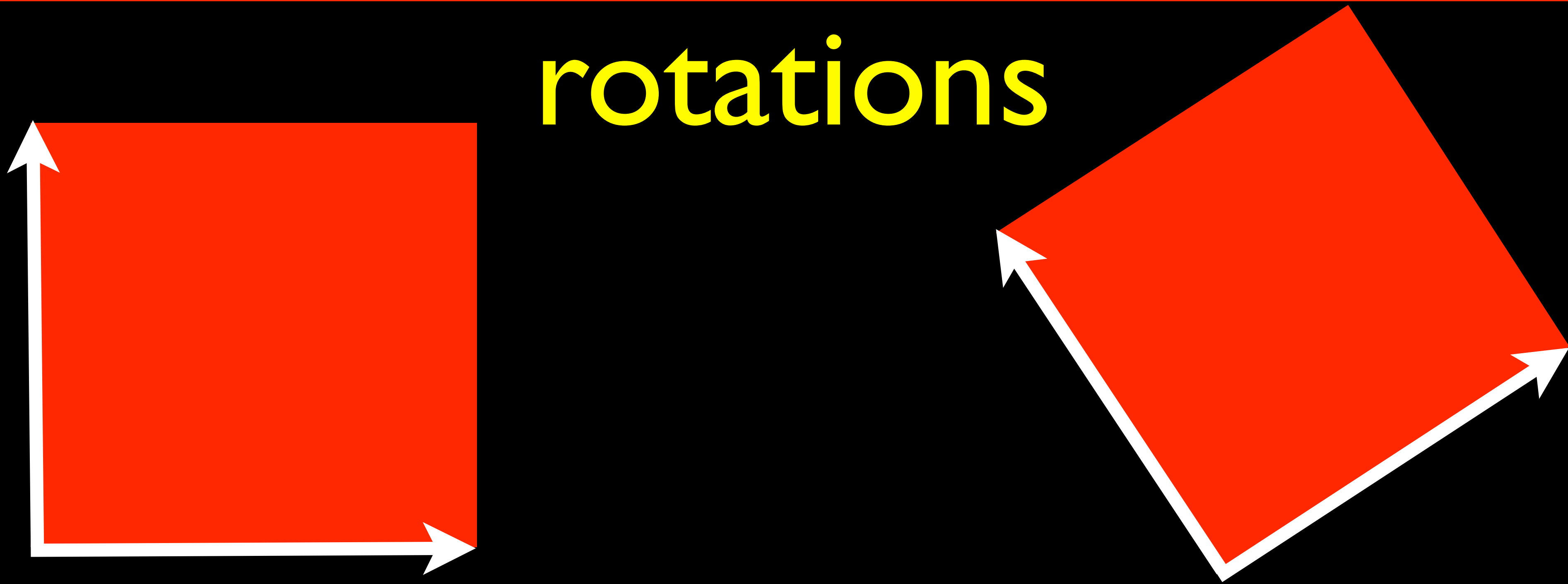
Example

What does the following transformation do?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

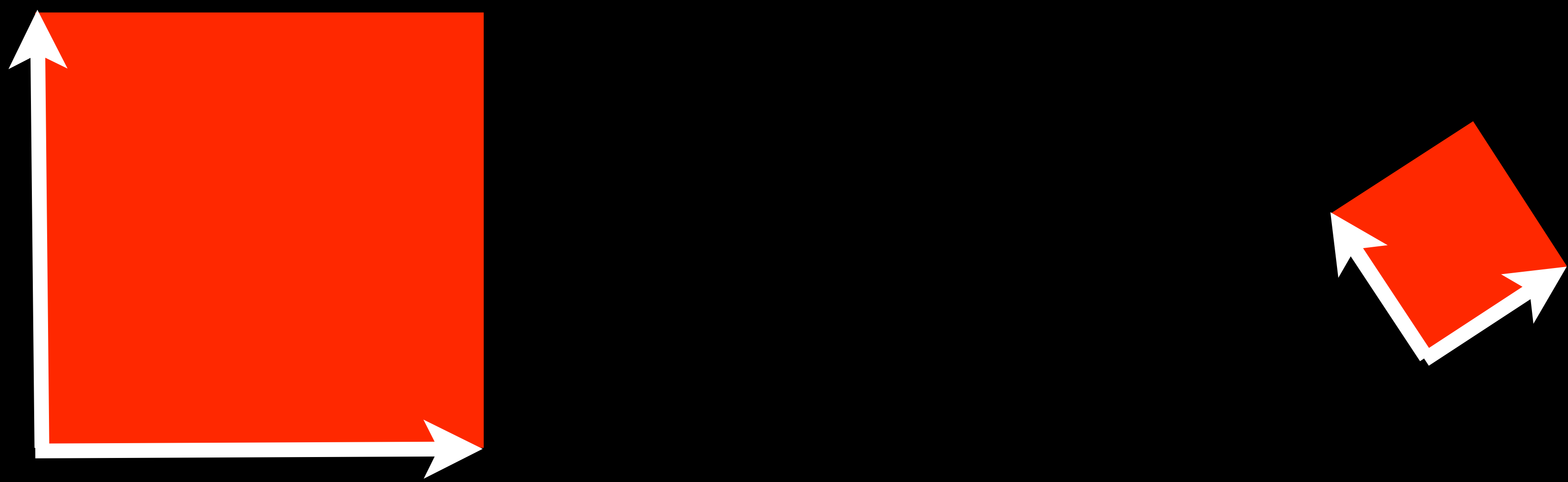
Examples of transformations

rotations



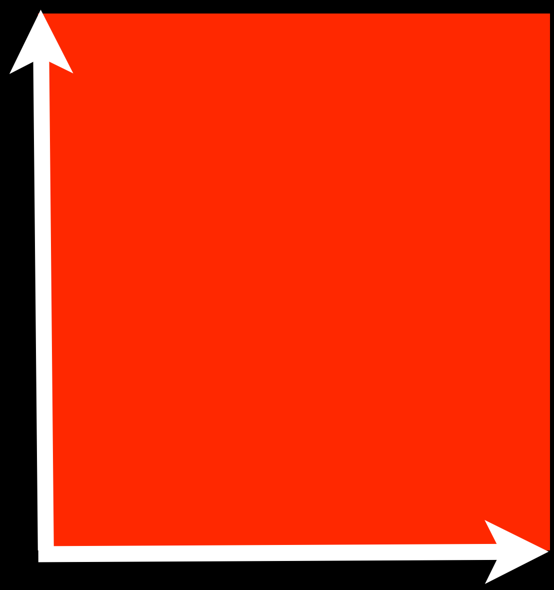
$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

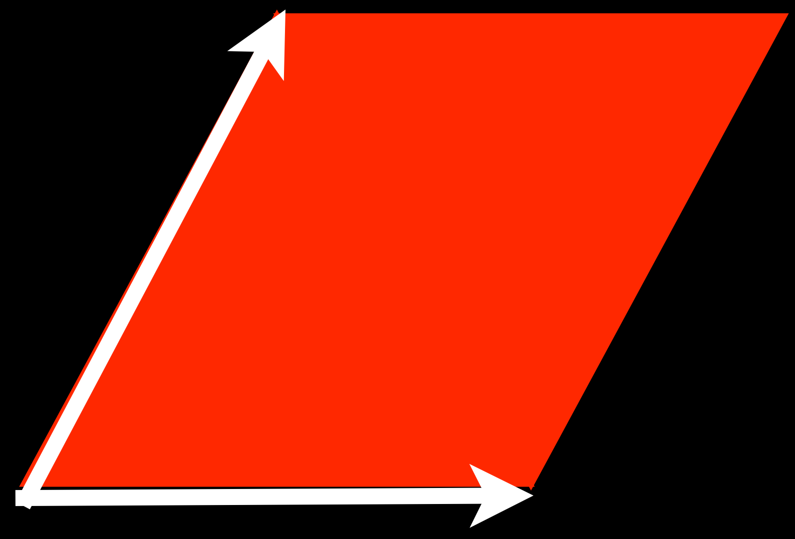
rotation-dilation



$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

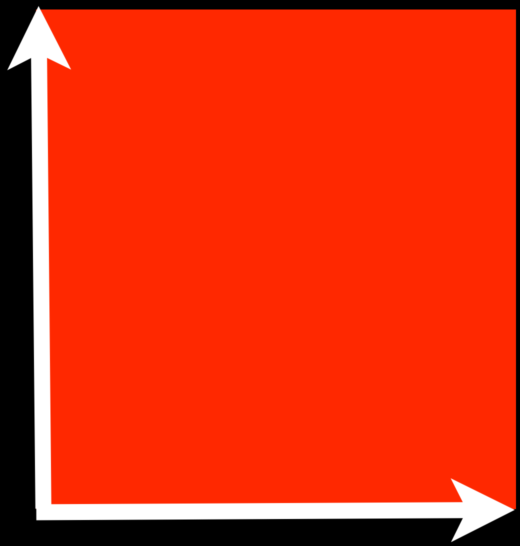
shears

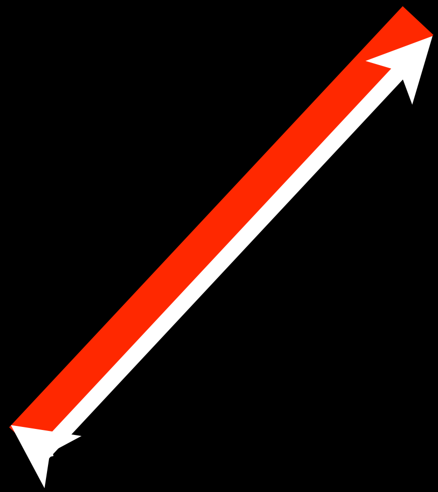




$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

projections

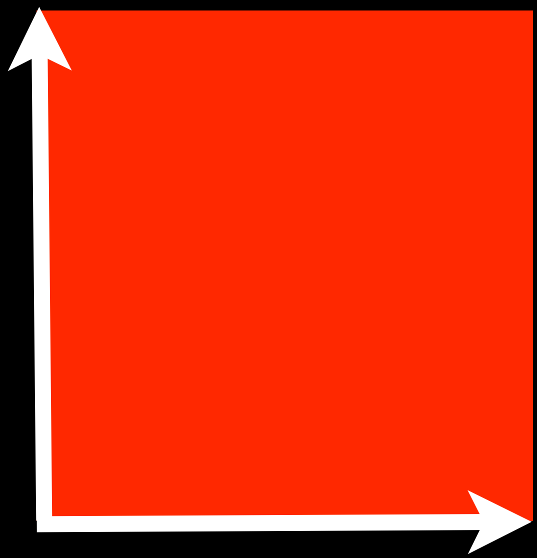


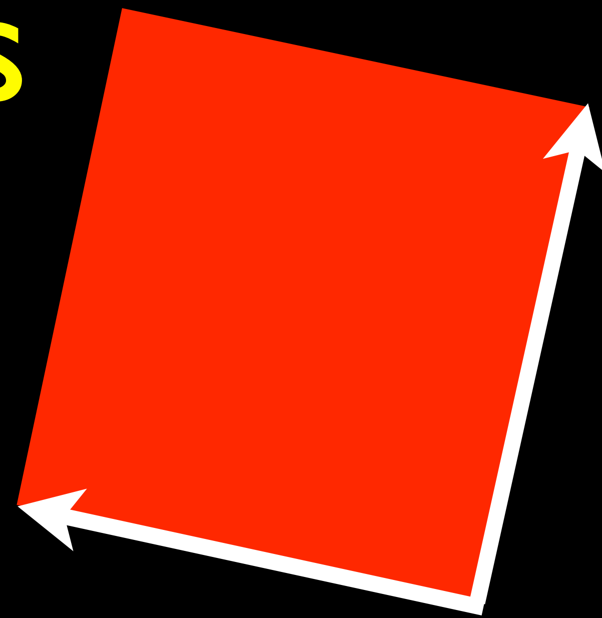


$$\begin{bmatrix} u_1 u_1 & u_2 u_1 \\ u_1 u_2 & u_2 u_2 \end{bmatrix}$$

Oops, the octomom
hype is gone.

reflections





$$\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

dilations

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

blackboard problem

Find the matrix of the transformation in R^4

which reflects at the xz plane then reflects at the yz plane.

Quiz coming up!


BRAD
PITT
MORGAN
FREEMAN
GLUTTONY
GREED
SLOTH
ENVY
WRATH
PRIDE
LUST
overlook columns
eight deadly sins, eight ways to die
Eight


JEFF BRIDGES
JOHN GOODMAN
THE
BIG LEBOW$KI

$$10^{100} =$$

100000000000000000000

00000000000000000000

00000000000000000000

00000000000000000000

00000000000000000000

= 1 gogool

What is the length of

$$\begin{bmatrix} 8 & -6 & 0 \\ 6 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{100} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The answer is ....

One googolplex

$$10^{10^{100}}$$

The name gogool was a term coined by Milton Sirotta (1929-1980), nephew of Edward Kasner (1878-1955). The googolplex number can not be written down in our universe (10^{80} atoms)

III. Basis

Linear independence

If

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$$

then, $c_1 = c_2 = \ldots = c_n = 0$

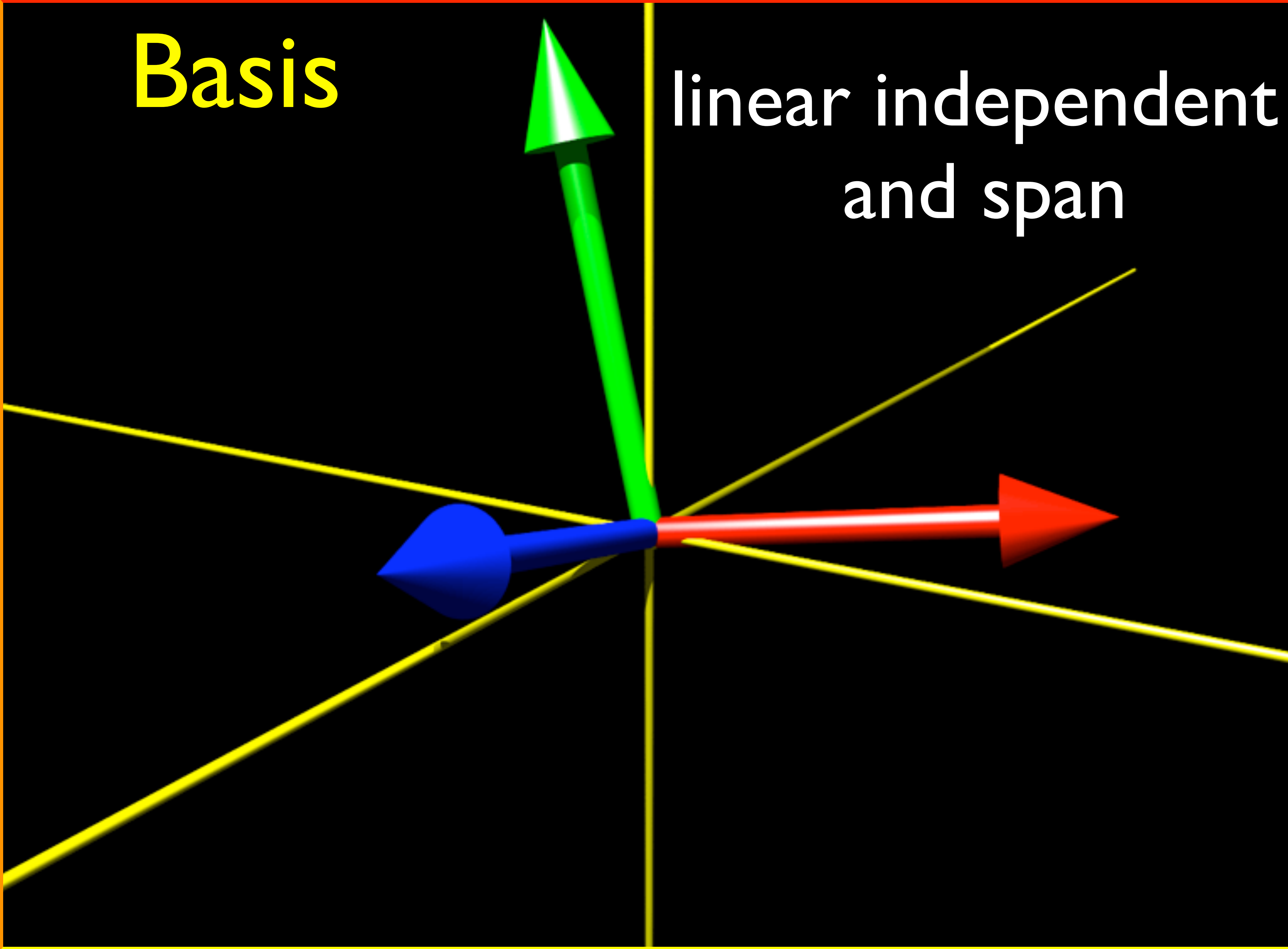
Spanning

every v in V can be written as

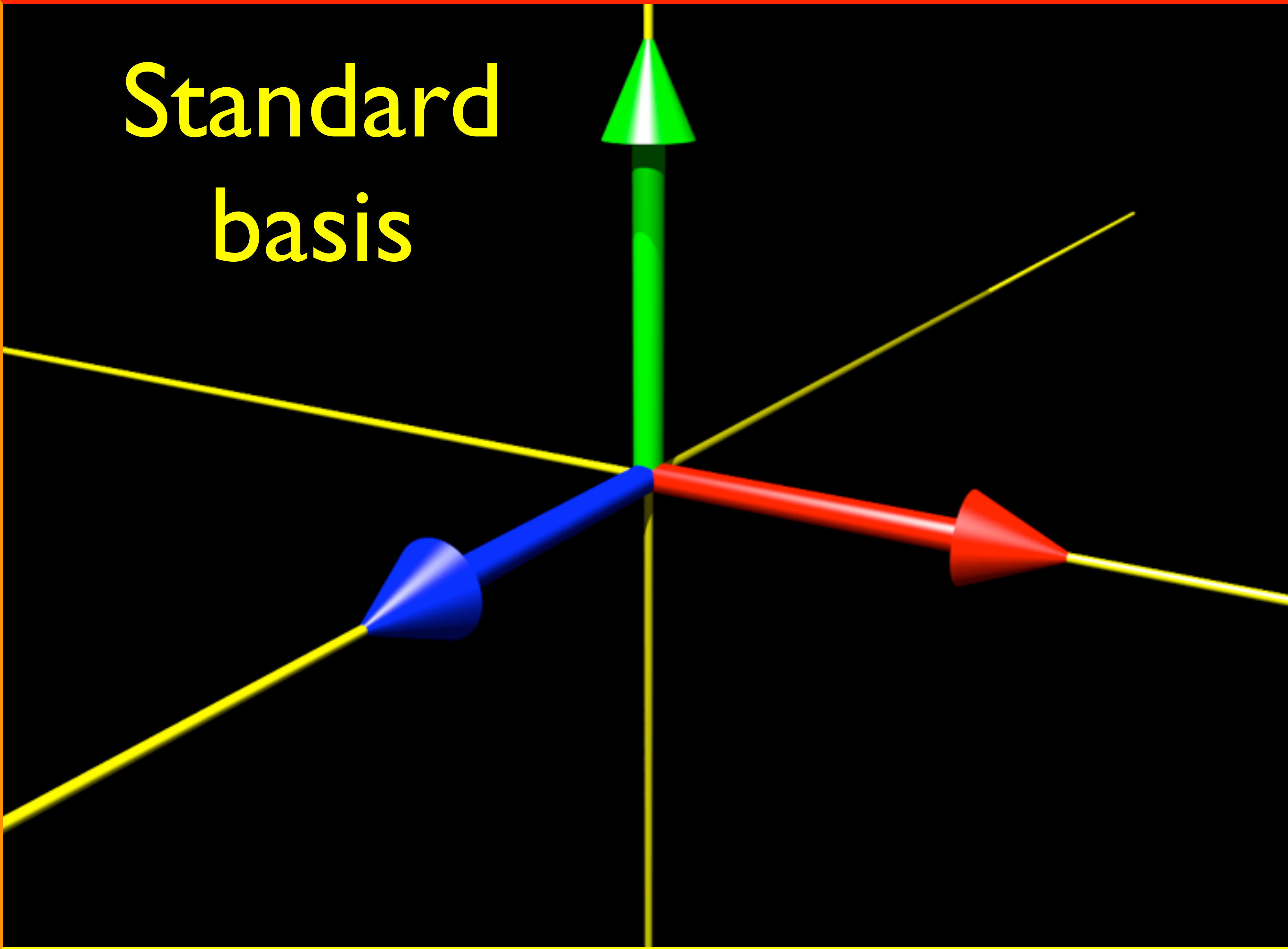
$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = v$$

Basis

linear independent and span



Standard basis

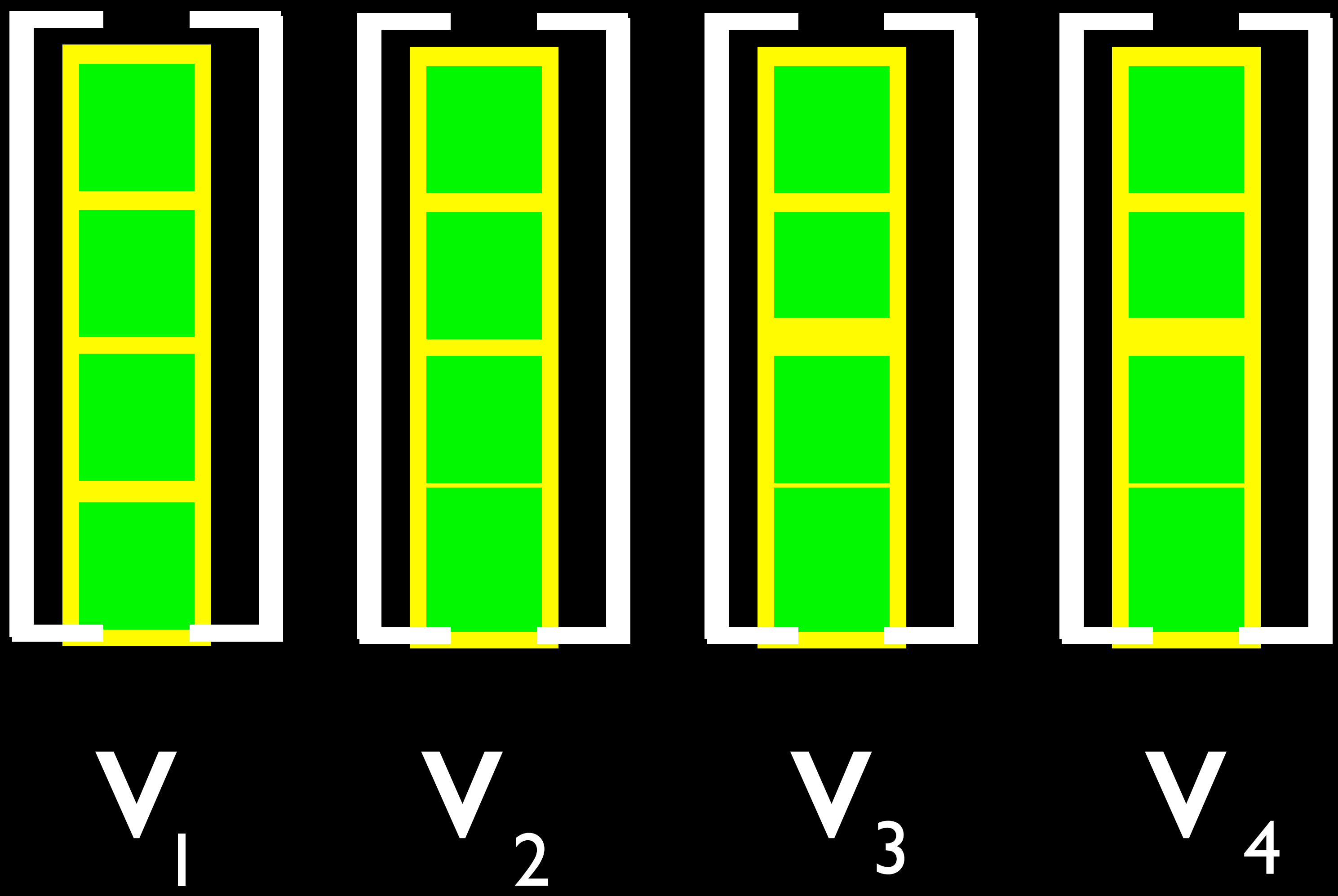


Standard basis

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$e_1 \qquad e_2 \qquad e_3 \qquad e_4$$

How do we check to have a basis?

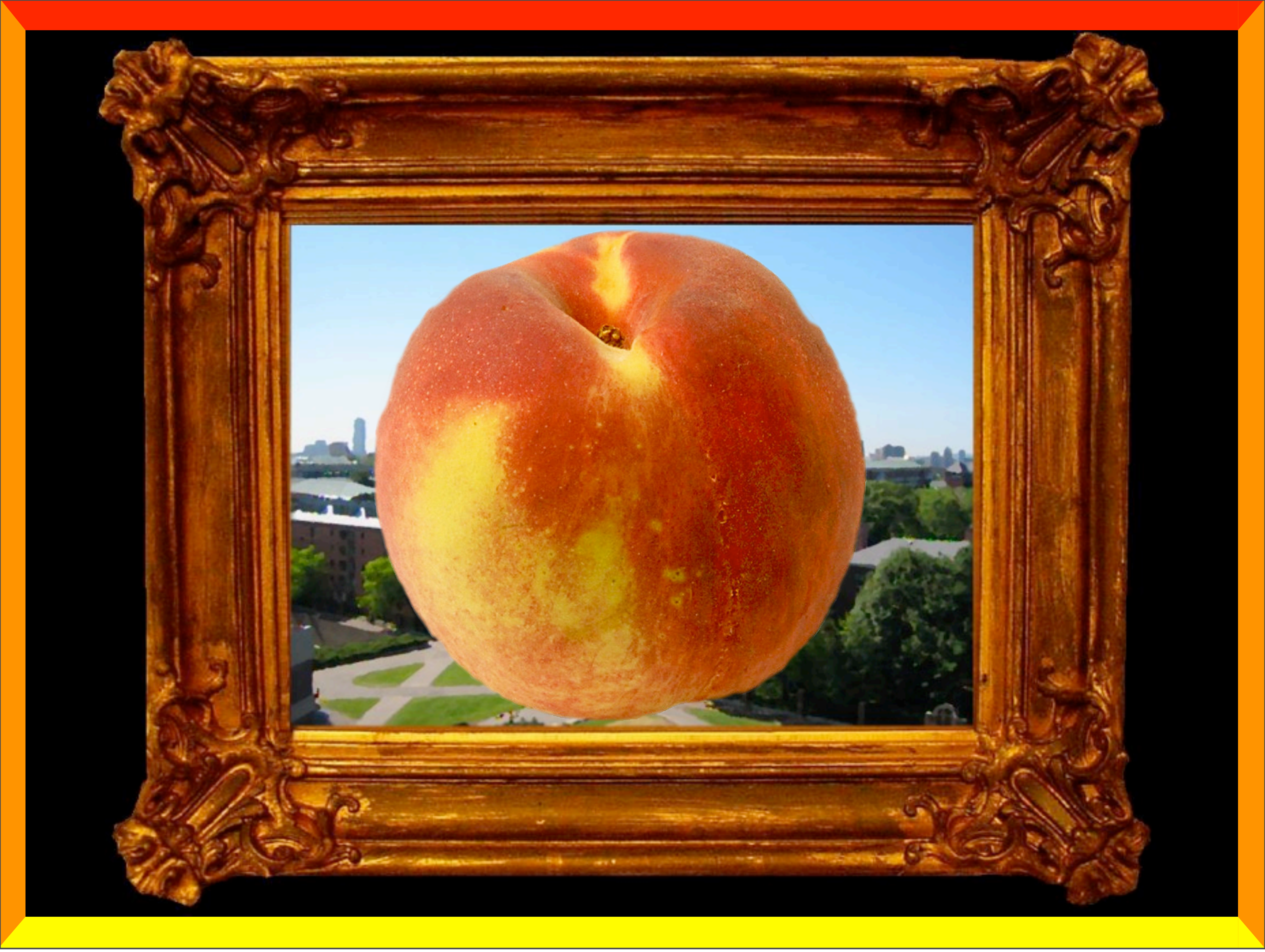


blackboard problem

Is this a basis?

$$v_1 = \begin{bmatrix} 8 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 8 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 8 \end{bmatrix}$$

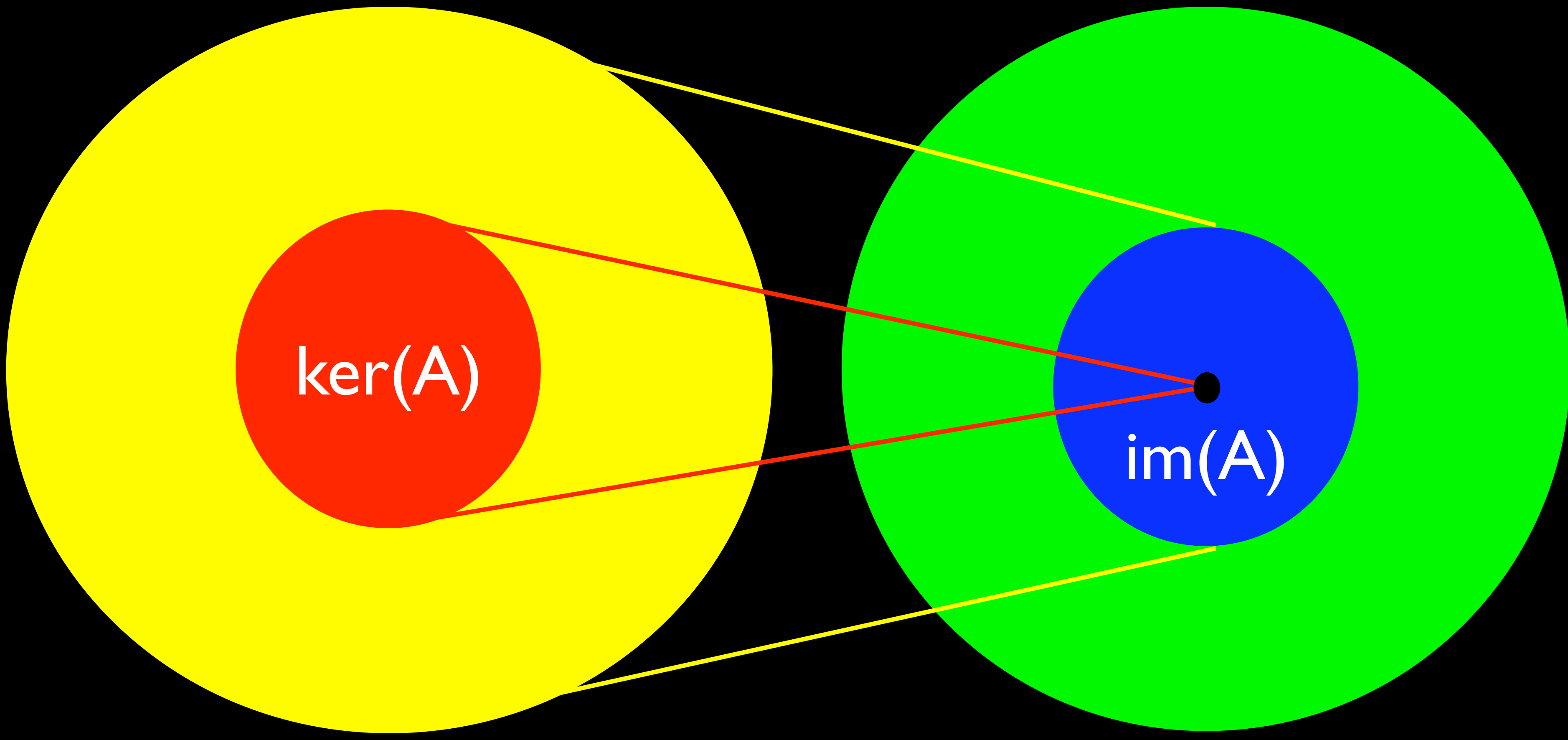
IV. Image and Kernel



Rene Magritte,
The son of man: 1963


Oliver Knill
Son of a peach, 2007

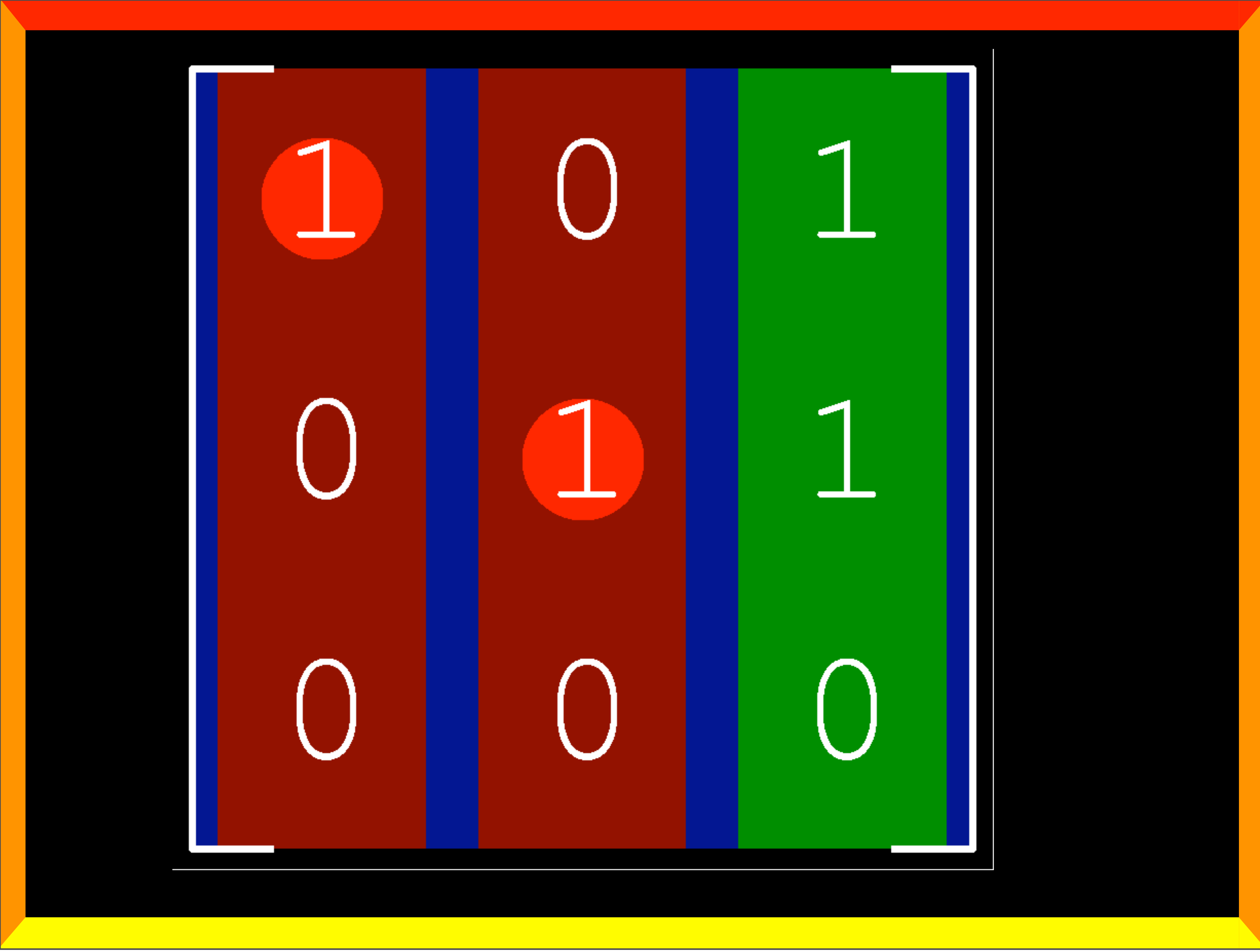
$$\text{im}(A) = \{ Ax \mid x \text{ in } V \}$$



$$\ker(A) = \{ x \mid Ax = 0 \}$$

How do we compute a basis for the image and kernel?

row reduce!


1 0 1
0 1 1
0 0 0

$$
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Dimension formula

rank nullety

$$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = m$$

fundamental theorem of linear algebra.

rank nulletly theorem

Pivot or not pivot, that is here the question

n

m

Computing the image:

The basis is the set of pivot columns.

Computing the kernel:

The basis is obtained by solving the linear system in row reduced echelon form and taking free variables.

blackboard problem

Image and kernel of X matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

XXX (2003) , movie won Taurus award for this stunt.
18 cameras filmed in Auburn CA at 800 feet bridge.

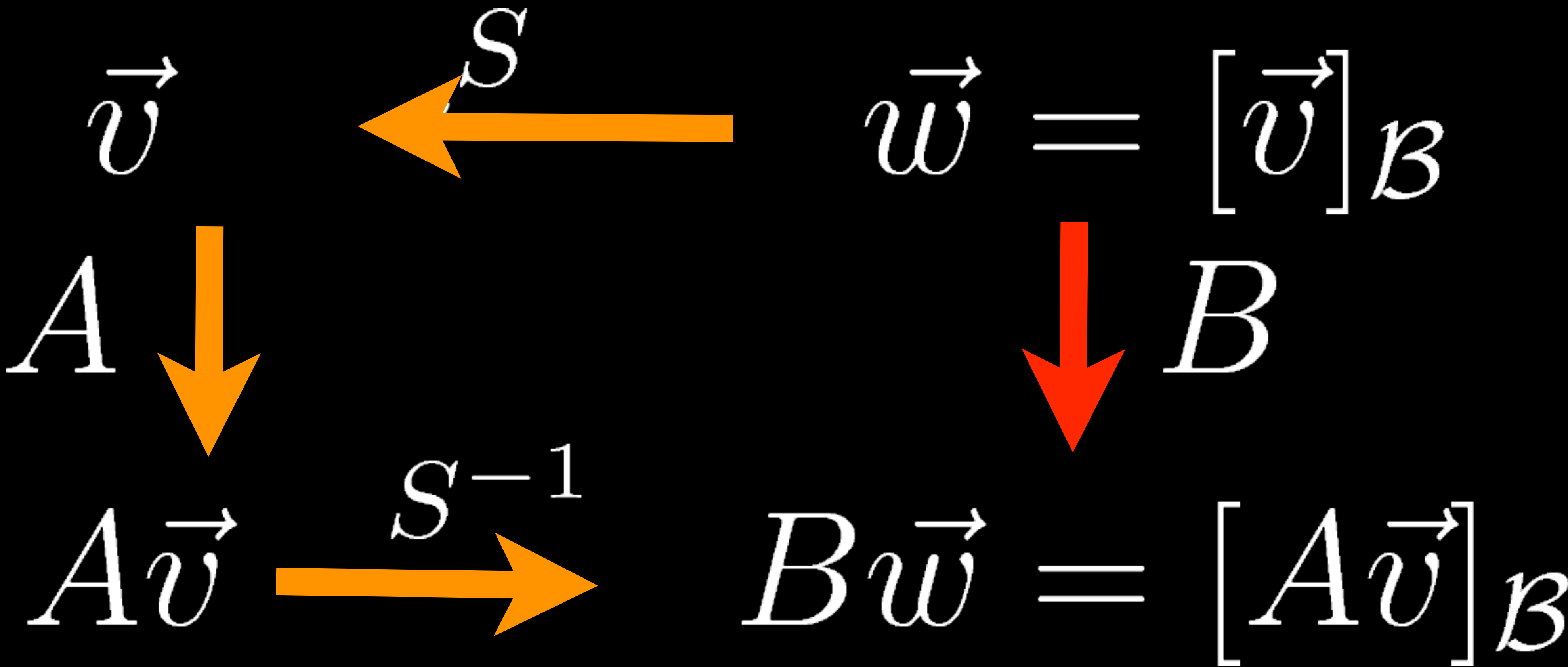
IV. Coordinates

v coordinates in standard basis

$$[v]_B = S^{-1} v$$

[v] coordinates in basis B

$$B = S^{-1} A S$$



Problem

$$B=\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$$

What are the B coordinates of v = $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

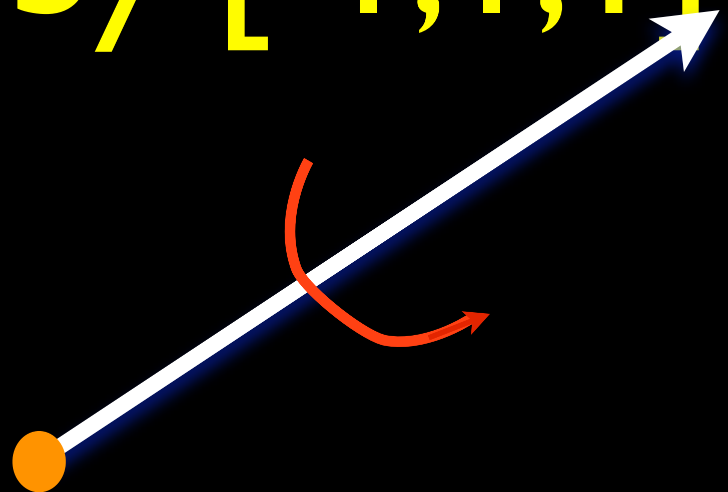
$$S^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_v^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

blackboard problem

Find the matrix A of the rotation by 90° about the axis spanned by [-1,1,1]



$$S = \begin{pmatrix} -1 & x & x \\ 1 & x & x \\ 1 & x & x \end{pmatrix}$$

$$B = S^{-1} A S$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A = S\ B\ S^{-1}$$

Similarity

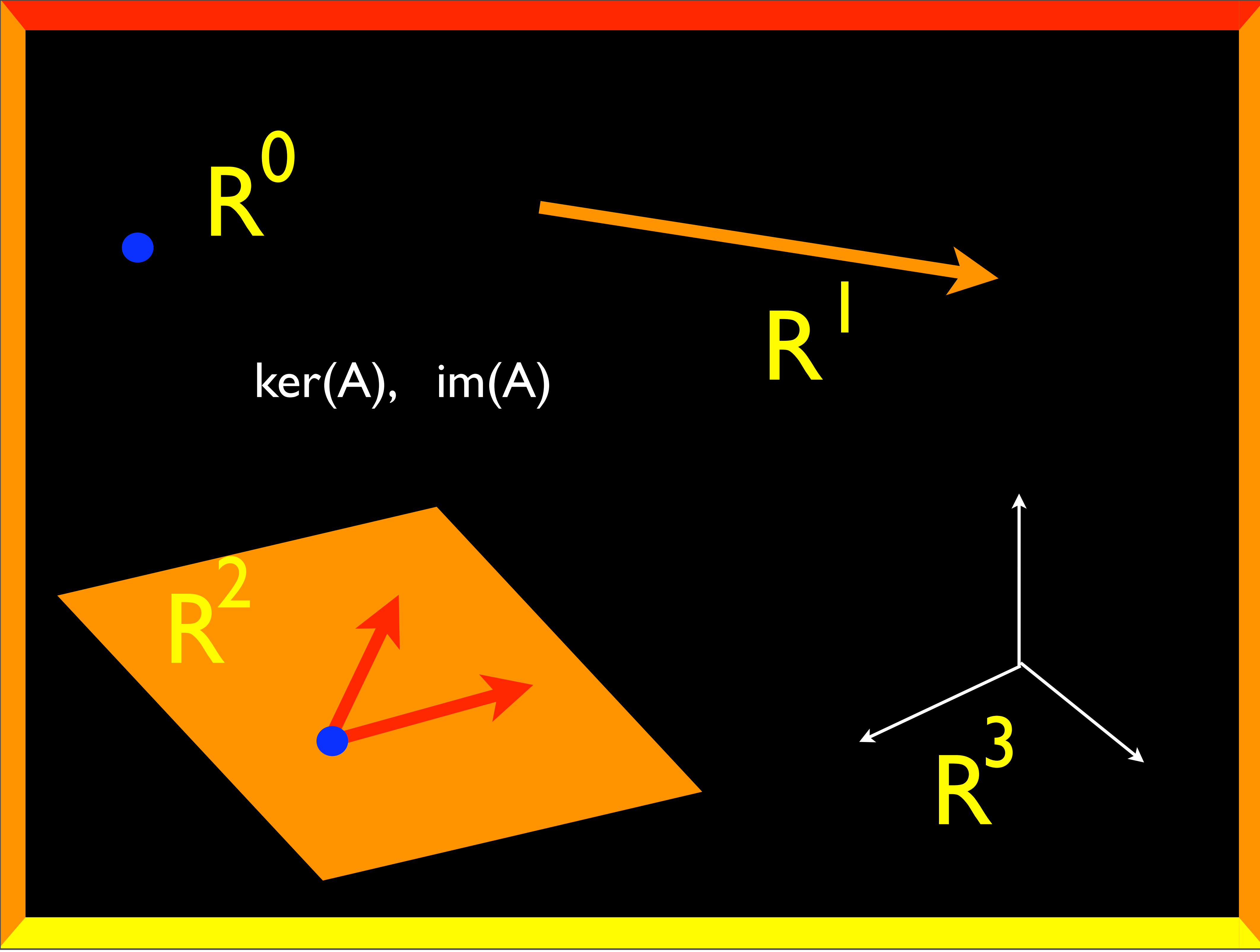
$$B = S^{-1} A S$$

The matrices A,B describe the same transformation!

The only matrix similar to the identity matrix is the identity matrix itself.

$$B^n = S^{-1} A^n S$$

V. Linear Spaces


R^0
R^1
ker(A), im(A)
R^2
R^3

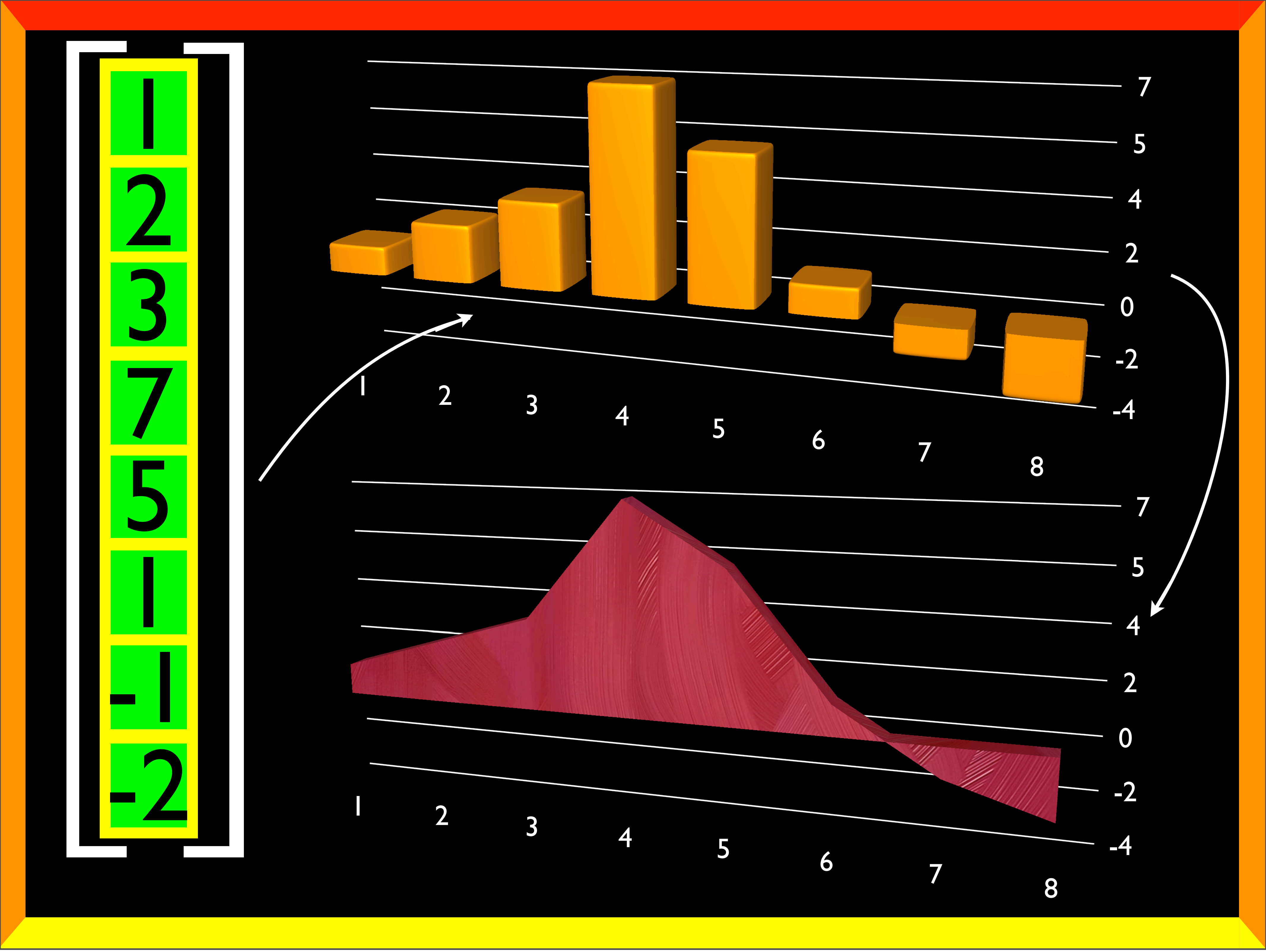
R^n = space of all maps from {1,2,...,n} to R

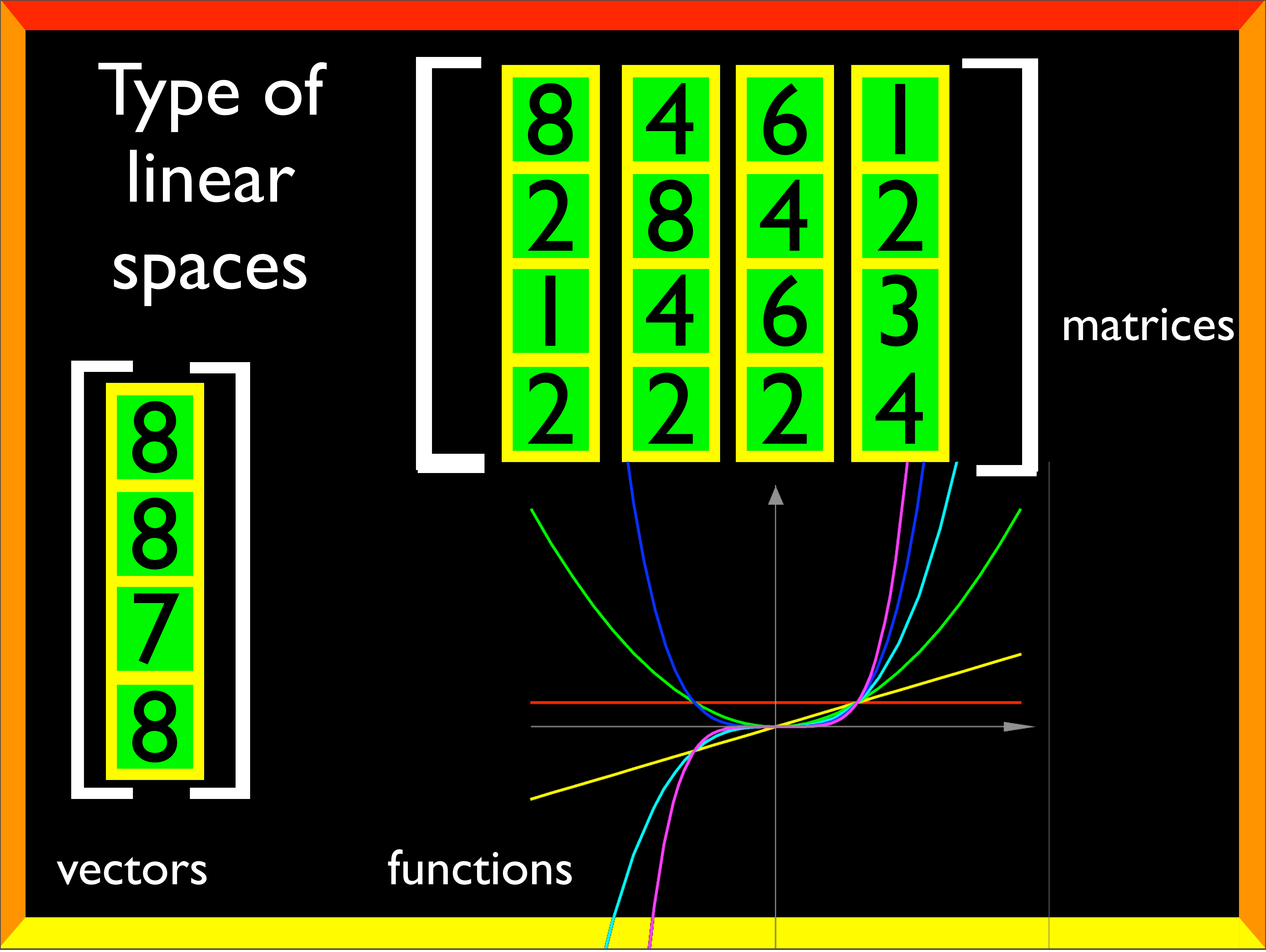
we write the map

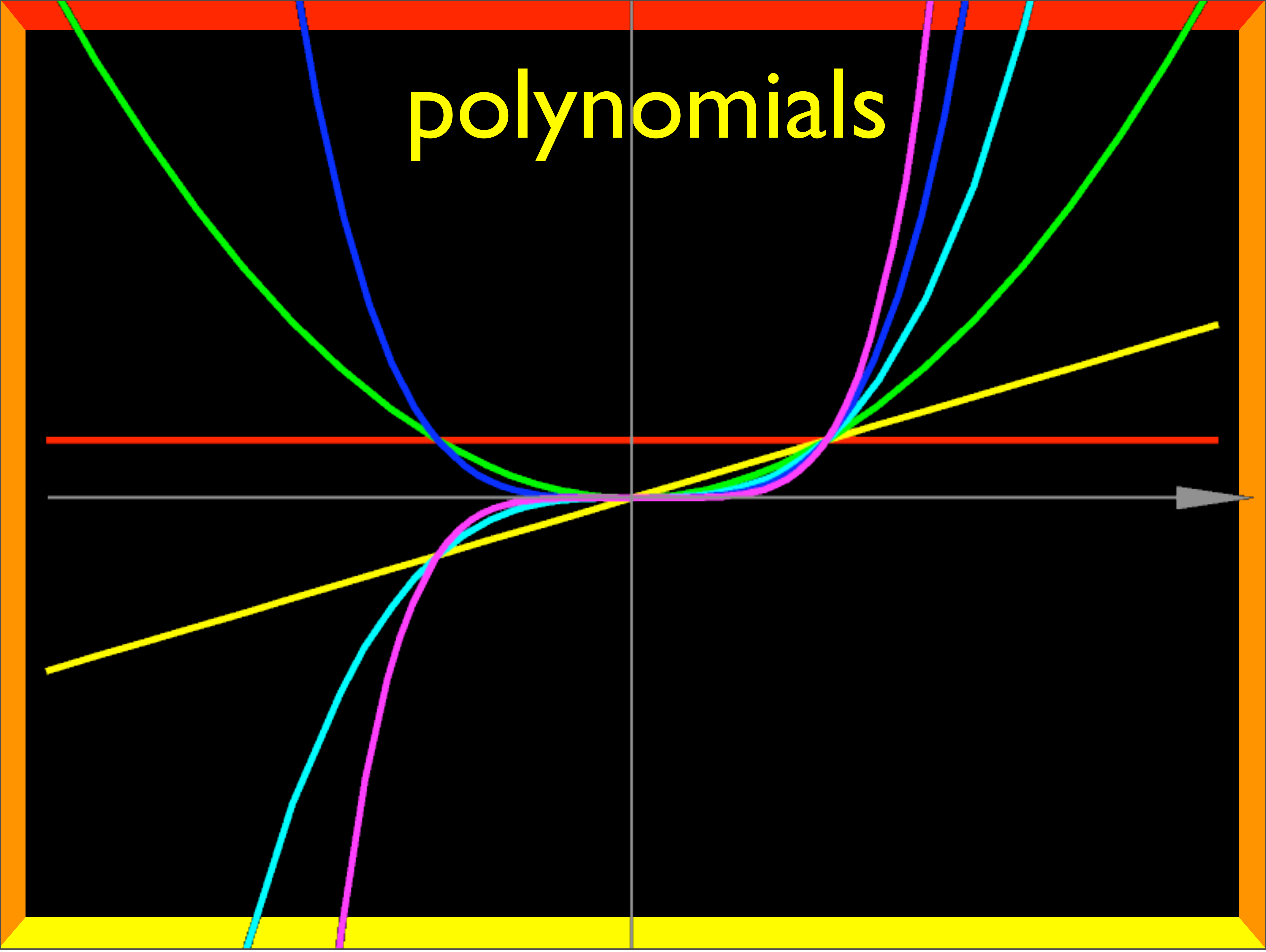
$\{1,2,3\} \longrightarrow \{5,7,-8\}$

as

$$\begin{bmatrix} 5 \\ 7 \\ -8 \end{bmatrix}$$


1
2
3
7
5
1
-1
-2
7
5
4
2
0
-2
-4
1
2
3
4
5
6
7
8
7
5
4
2
0
-2
-4
1
2
3
4
5
6
7
8


Type of linear spaces
8 4 6 1
2 8 4 2
1 4 6 3
2 2 2 4
matrices
8
8
7
8
vectors
functions


polynomials

solutions to differential equations

Solutions to linear differential equations are linear spaces.

Example:

$$X=\{ f \text{ in } C^{\infty}(R) \mid f''(x)=-f(x)\}$$

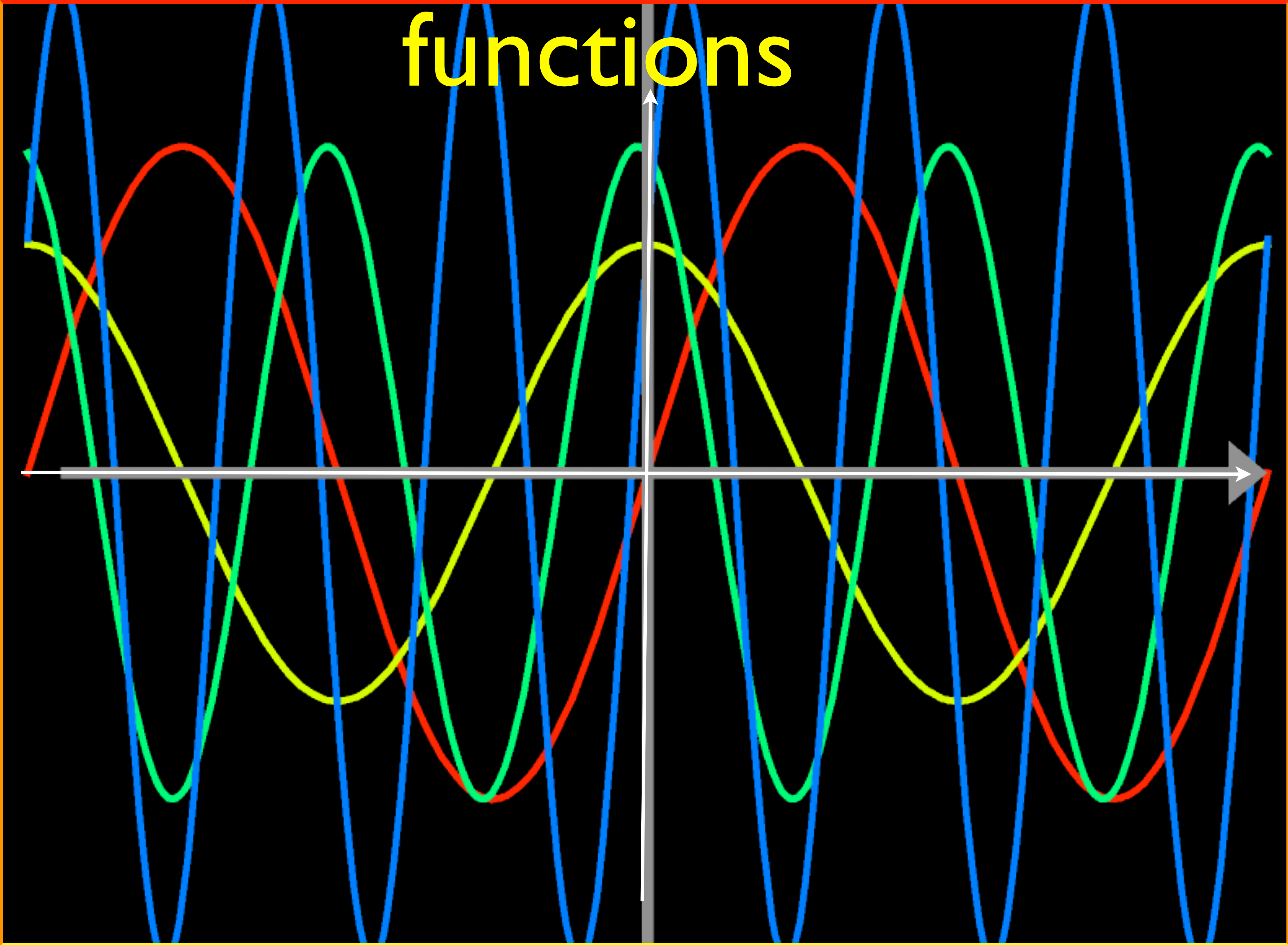
matrices

$$\begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}$$

$$-3 \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

functions



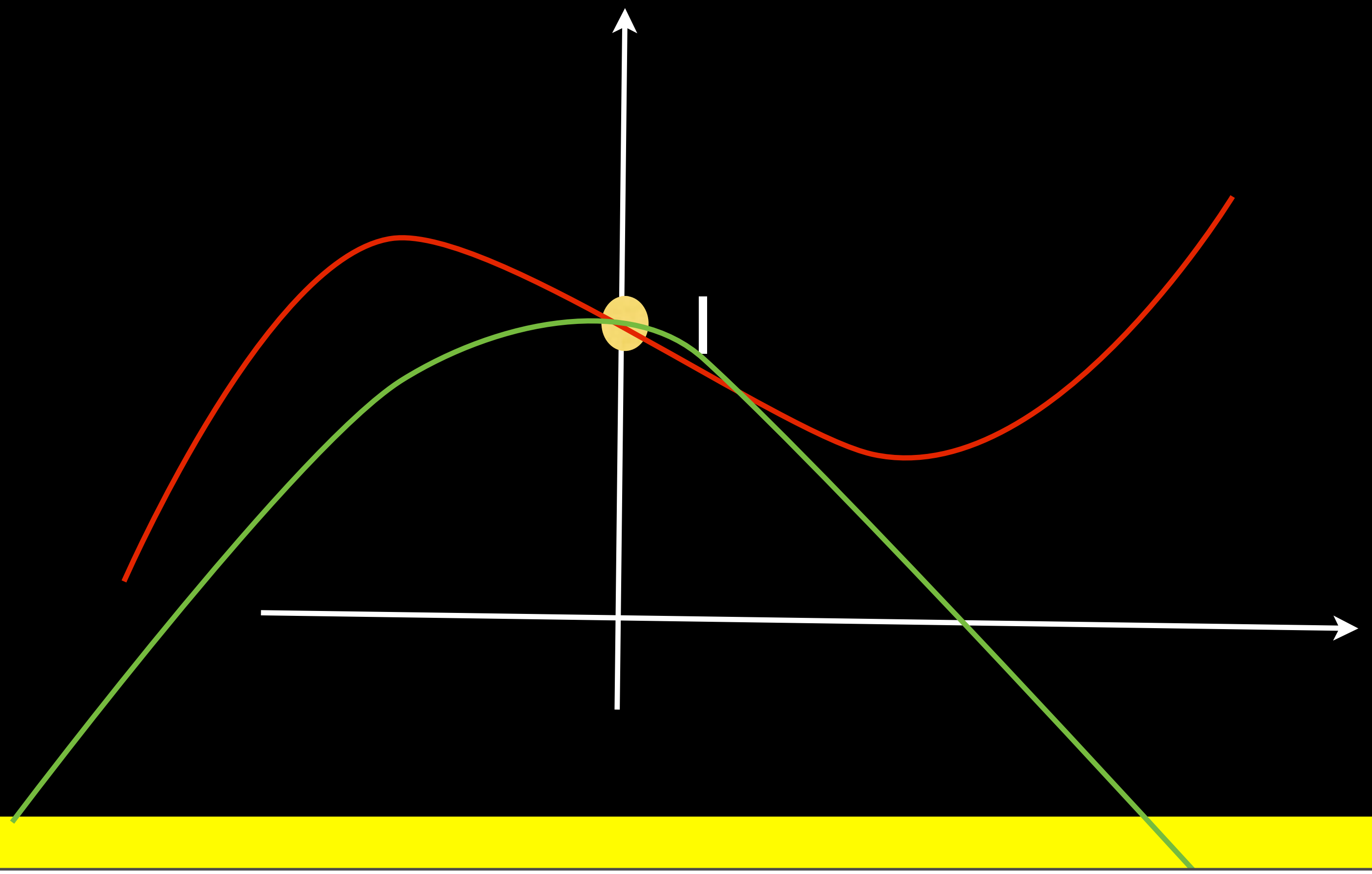
blackboard problem

Is this a linear space?

$$X = \{f \text{ in } P_2 \mid f''(0) = 1\}$$

Is this a linear space?

$$X = \{ f \text{ in } P_3 \mid f(0) = 1 \}$$



Is this a linear space?

$$X = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^2 = I_2 \right\}$$

The end