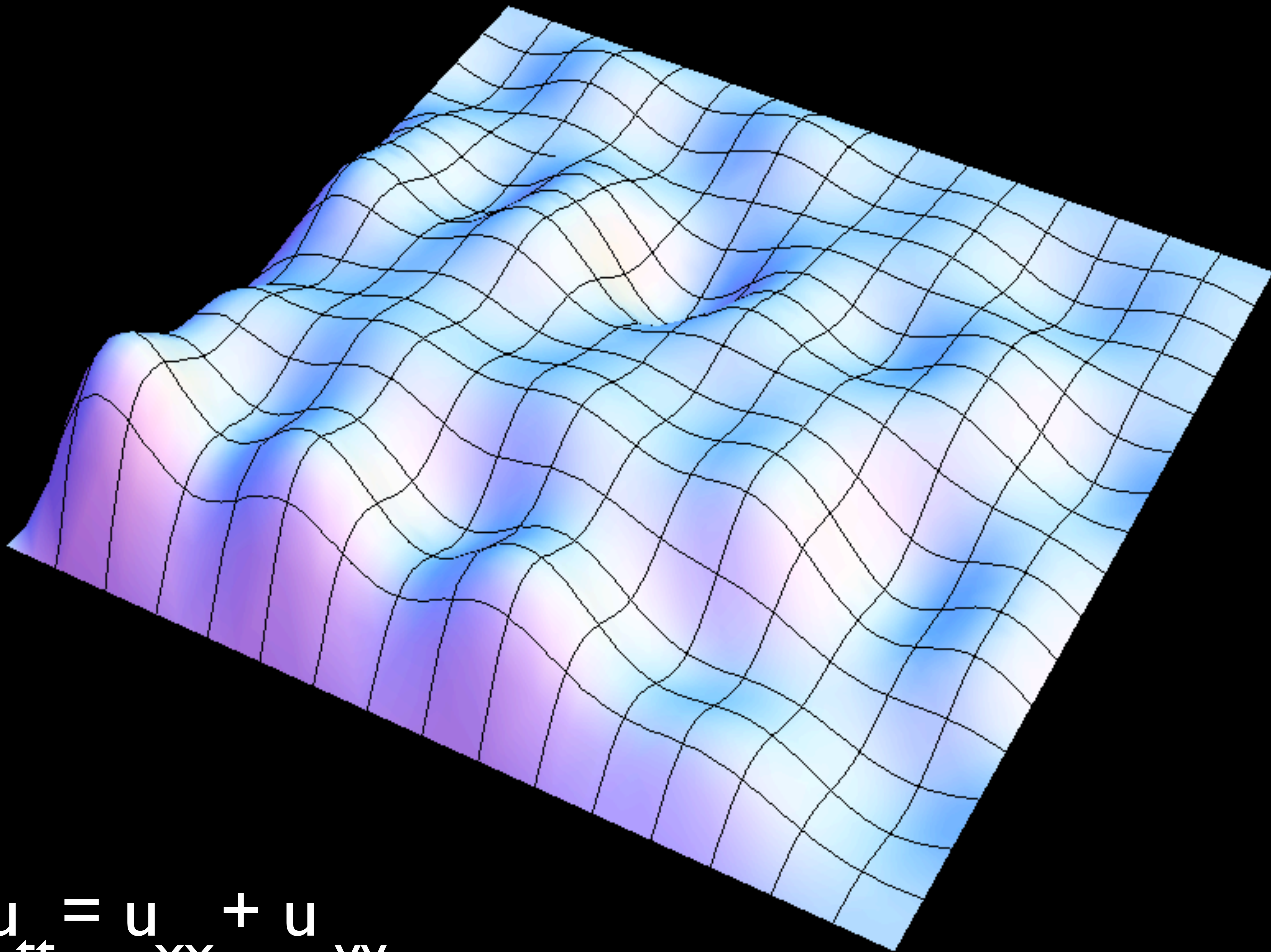
Math21b

Final Review

Spring 2009

Oliver Knill, May 12, 2009



$$u_{tt} = u_{xx} + u_{yy}$$

We focus on the material after the 2. midterm, but repeat now some key points

Plan

- Early key points: Ax=b, det, fit, diagonalization,
- Discrete dynamical systems $x(t+1) = A\,x(t)$
- Linear differential equations $\dot{x} = A\,x$
- Nonlinear systems $\dot{x} = f(x,y), \dot{y}=g(x,y)$
- Operator methods $f''+af'+bf=g$
- Fourier theory $f(x) = \sum_n b_n \sin(nx)$
- Partial differential equations $u_{tt} = u_{xx} + u_{xxxx}$

PART I

Key points from before second midterm

Linear equations

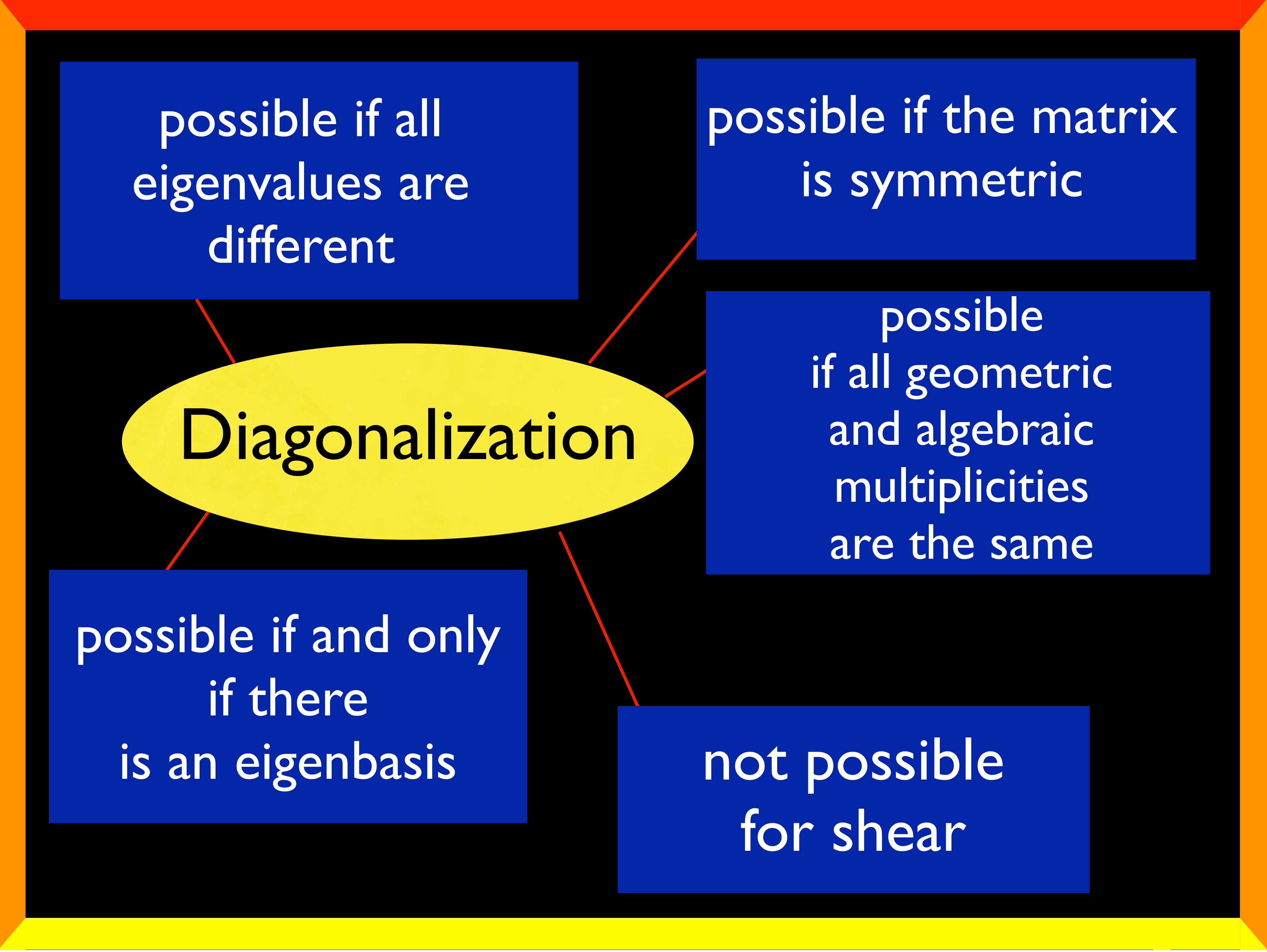
- $A x = b$
- $x = A^{-1} b$
- row reduce $[A| b]$
- Solutions are $x_0 + \ker(A)$
- Least square solution
- consistent: have solution

find the general solution of:

$$x+y+z+v+w=6$$

$$2x-y+5z-v+8w=15$$

$$x+y+z+v+w=6$$
$$2x-y+5z-v+8w=15$$


Diagonalization
possible if all eigenvalues are different
possible if the matrix is symmetric
possible if all geometric and algebraic multiplicities are the same
possible if and only if there is an eigenbasis
not possible for shear

can we diagonalize?

1	2	3	0	0	0	0
2	8	2	0	0	0	0
3	2	9	0	0	0	0
0	0	0	1	0	0	0
0	0	0	4	6	0	0
0	0	0	9	10	11	0
0	0	0	13	14	15	16

Projection

$A A^T$

if columns orthogonal

$A (A^T A)^{-1} A^T$

$x = (A^T A)^{-1} A^T$

least square solution

$P^2 = P$

$P v = (u.v)\, v$

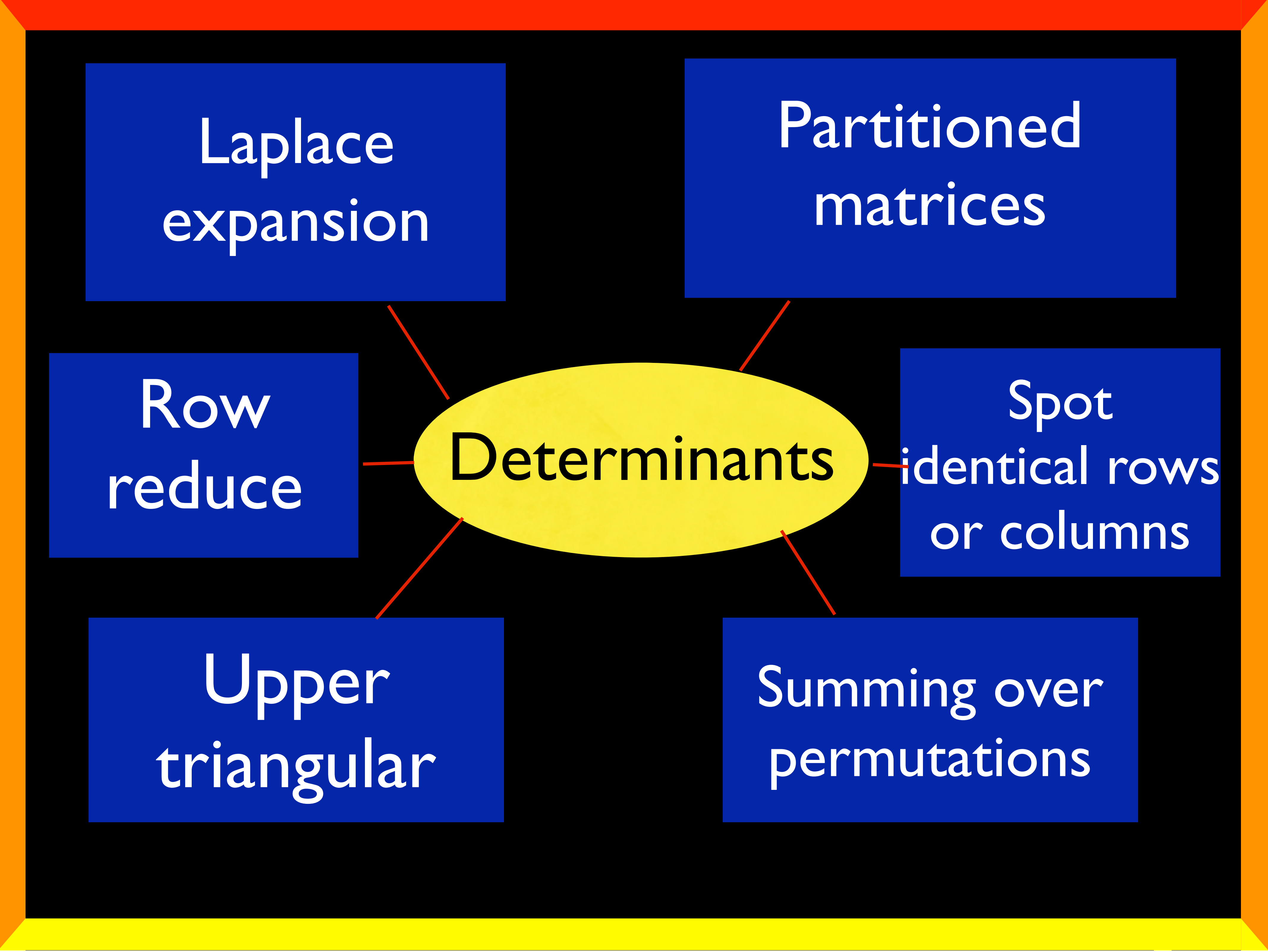
onto one dimensional line

Which one is invertible if A is a 3x2 matrix with orthonormal columns?

$A^T A$

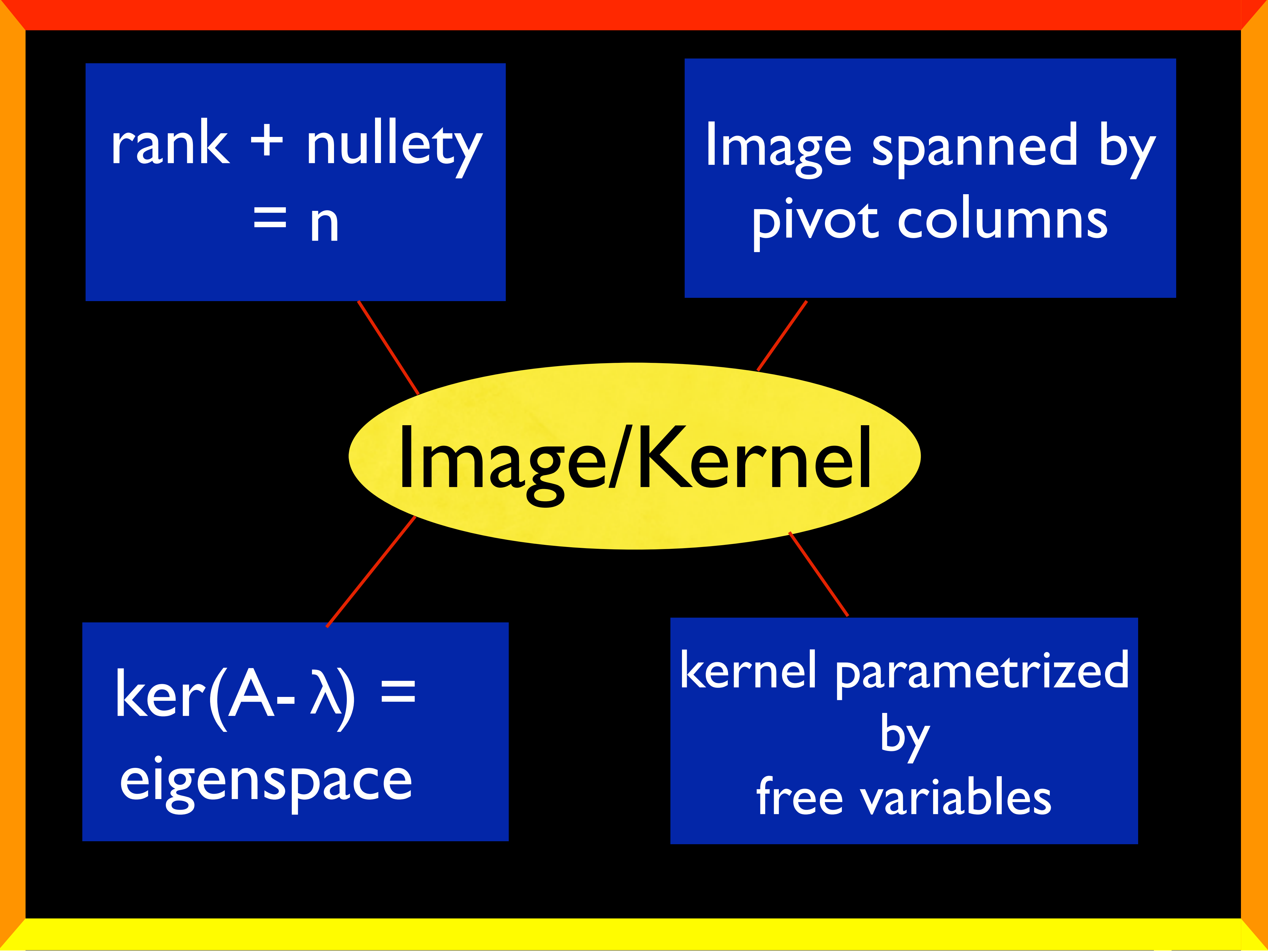
or

$A A^T$


Laplace expansion
Partitioned matrices
Row reduce
Determinants
Spot identical rows or columns
Upper triangular
Summing over permutations

problem

$$\det \begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 7 & 8 & 9 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 9 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 & 15 & 16 \end{pmatrix} = ?$$


rank + nullety = n
Image spanned by pivot columns
Image/Kernel
ker(A- λ) = eigenspace
kernel parametrized by free variables

Matrices

n x m matrix
n rows and m columns

Columns:
image of basis vectors

$(AB)^{-1} = B^{-1} A^{-1}$

$(AB)^{T} = B^{T} A^{T}$

$A B \neq B A$ in general

$B = S^{-1} A S$

similarity

$z = a + i\, b$

$= r\, e^{i\theta}$

$z = r \cos(t) + i\, r \sin(t)$

Euler formula

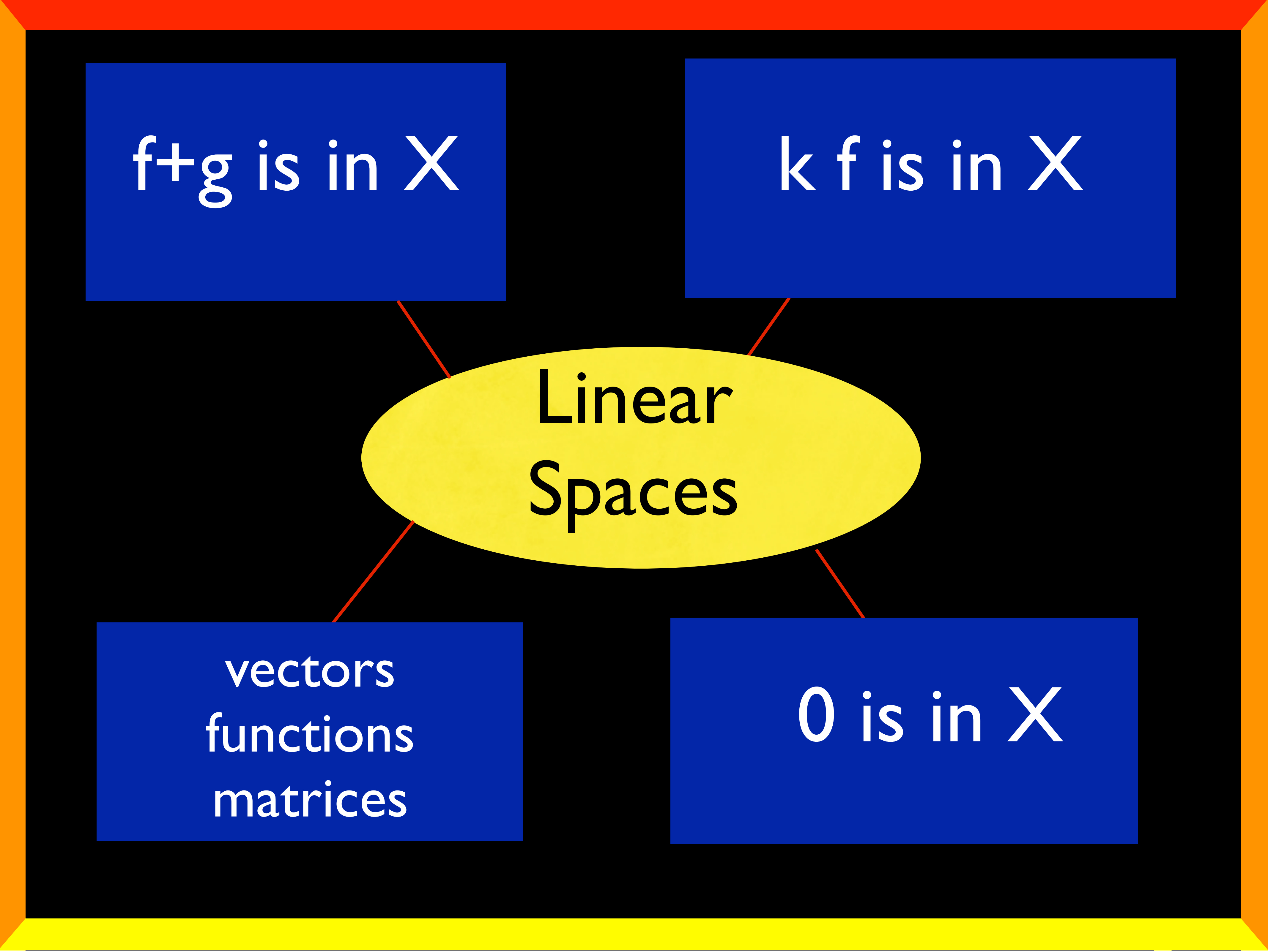
Complex Numbers

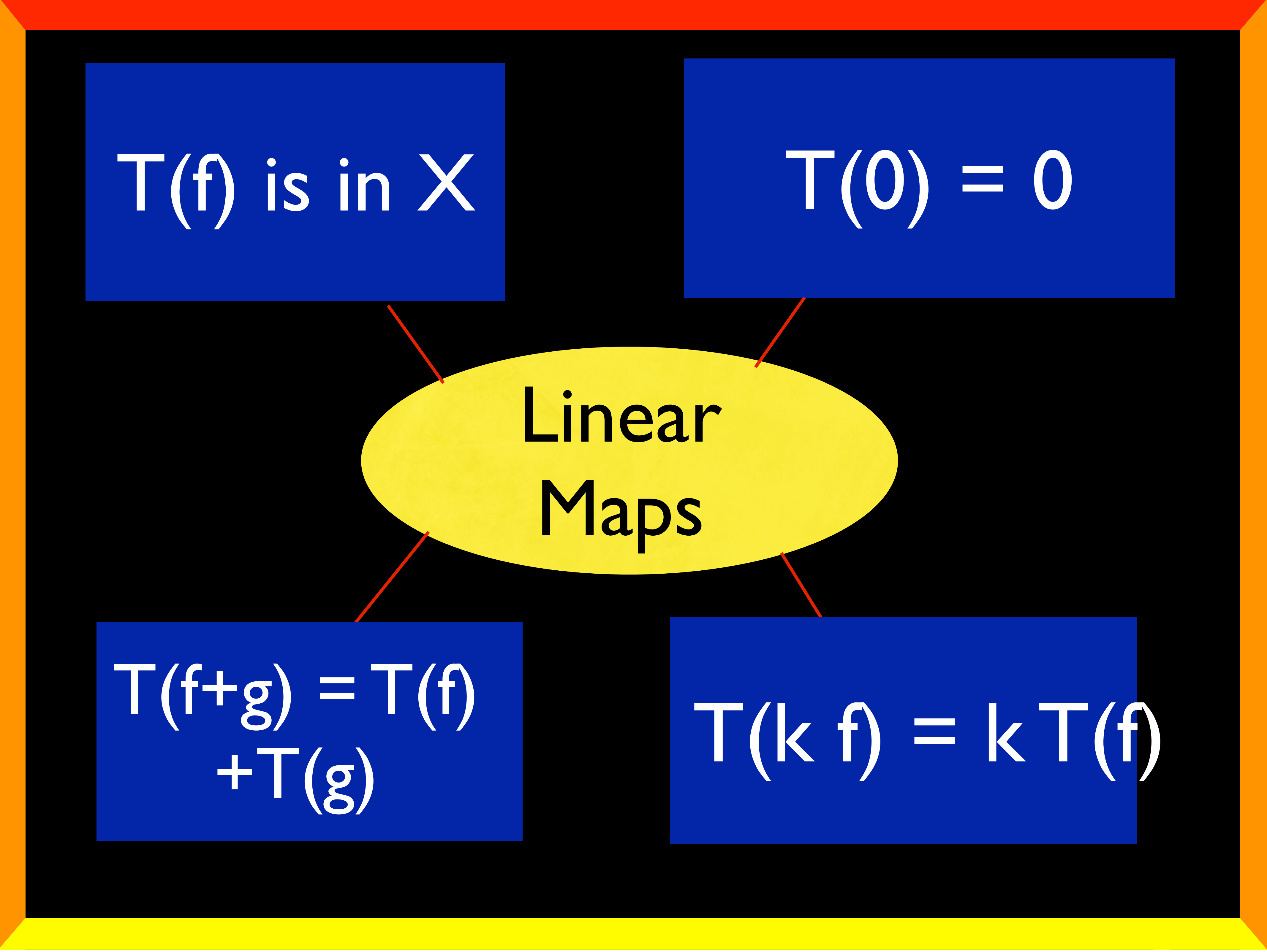
fundamental theorem of algebra

$f(\lambda) = \det(A-\lambda) =$

$(\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$

$p(D) = (\lambda_1 - D) \cdots (\lambda_n - D)$


f+g is in X
k f is in X
Linear Spaces
vectors
functions
matrices
0 is in X


T(f) is in X
T(0) = 0
Linear Maps
T(f+g) = T(f) +T(g)
T(k f) = k T(f)

problem

which of the following are linear spaces?

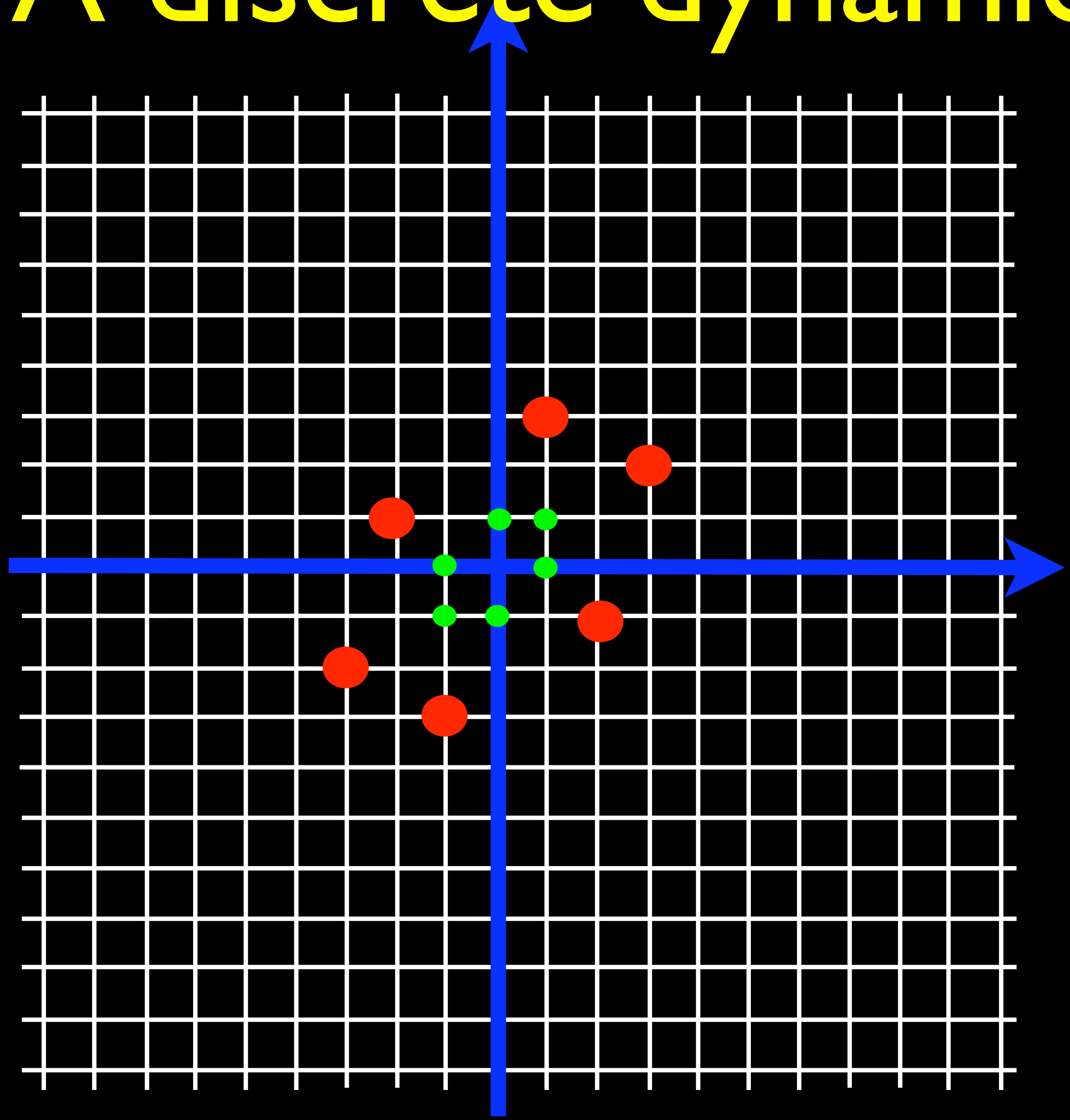
- Smooth 2π-periodic functions with $\int_0^{2\pi} f(x)\, dx = 0$.
- $\{ f \in C^\infty(\mathbf{R}) \mid f(10) = 1 \}$
- Smooth 2π-periodic functions satisfying $f'(0) = 0$.
- 2×2 matrices satisfying $\mathrm{tr}(A) = 0$.
- 2×2 matrices satisfying $\det(A) = 0$.

- $T(f)(x) = x^2 f(x).$
- $T(f)(x) = f(1)^2 + f(x).$
- $T(f)(x) = f'(x).$
- $T(f)(x) = f(x)f'(x).$
- $T(f)(x) = x + f(x).$

which T are linear transformations?

Discrete dynamical systems

A discrete dynamical system

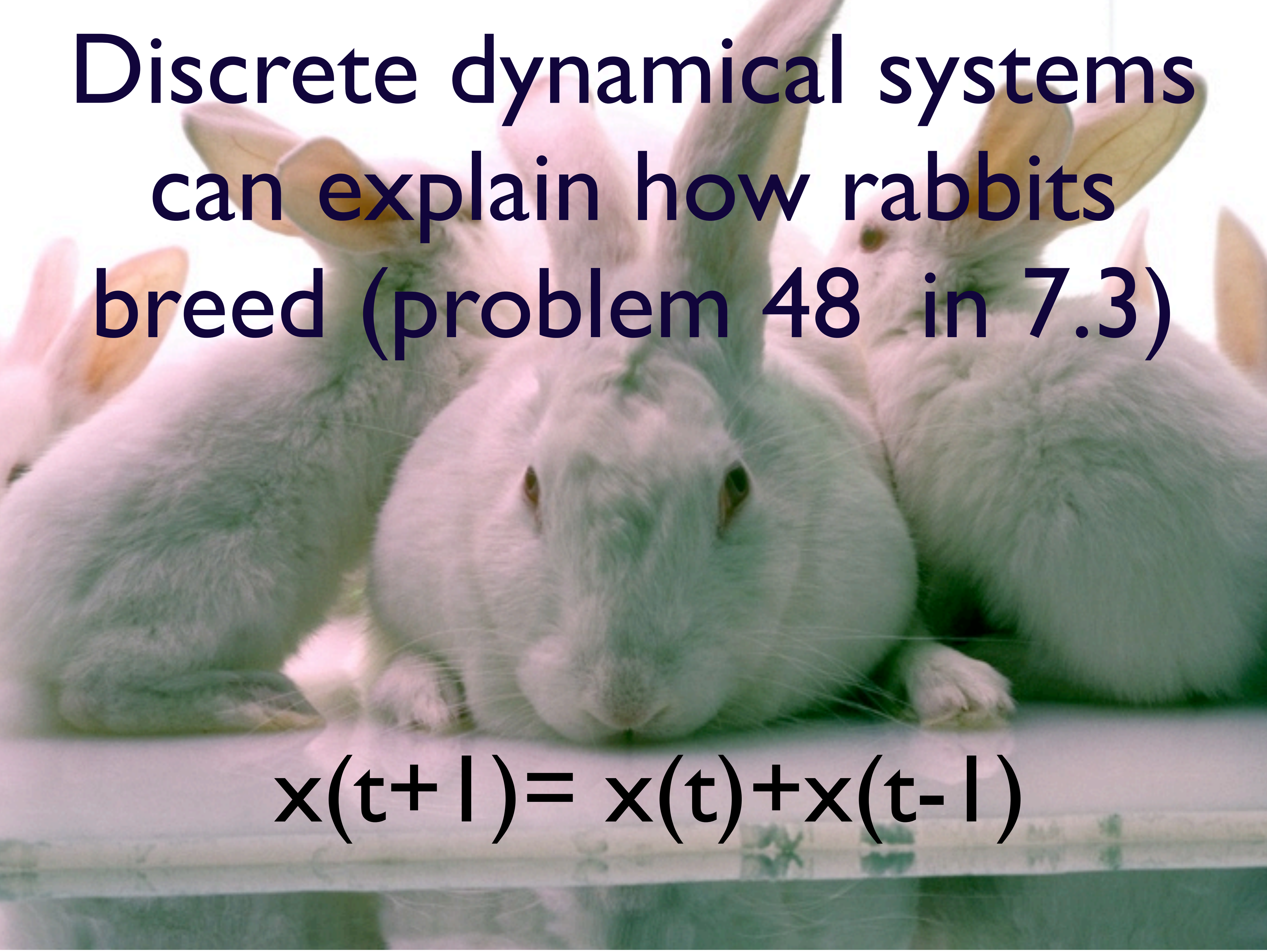


$x(t+1) = x(t) - y(t)$

$y(t+1) = x(t)$

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Discrete dynamical systems can explain how rabbits breed (problem 48 in 7.3)



$$x(t+1) = x(t) + x(t-1)$$

There is a problem in the book, which describes how Lilac bushes grow (problem 54 in 7.1)

here is a story from 2007 about panda love:

problem

Problem

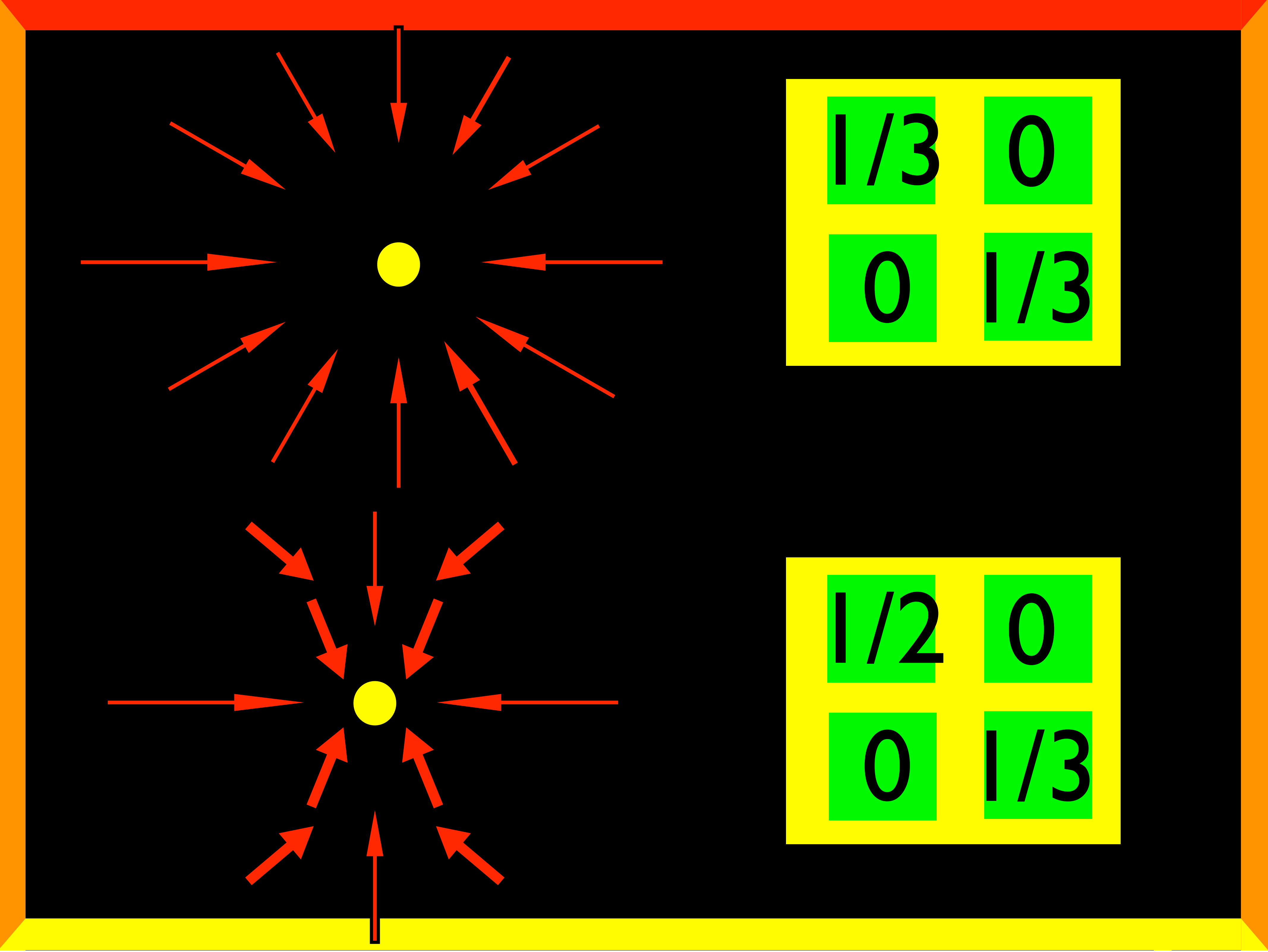
x(n) panda population at time n

$$x(n+1) = 2x(n) + y(n)$$
$$y(n+1) = x(n) + 2y(n)$$

Find a closed formula for x(n) and y(n) if $x(0)=6$, $y(0)=2$

$x(n+1) = 2x(n) + y(n)$ $y(n+1) = x(n) + 2y(n)$ $x(0)=6, y(0)=2$

Stability of discrete dynamical systems



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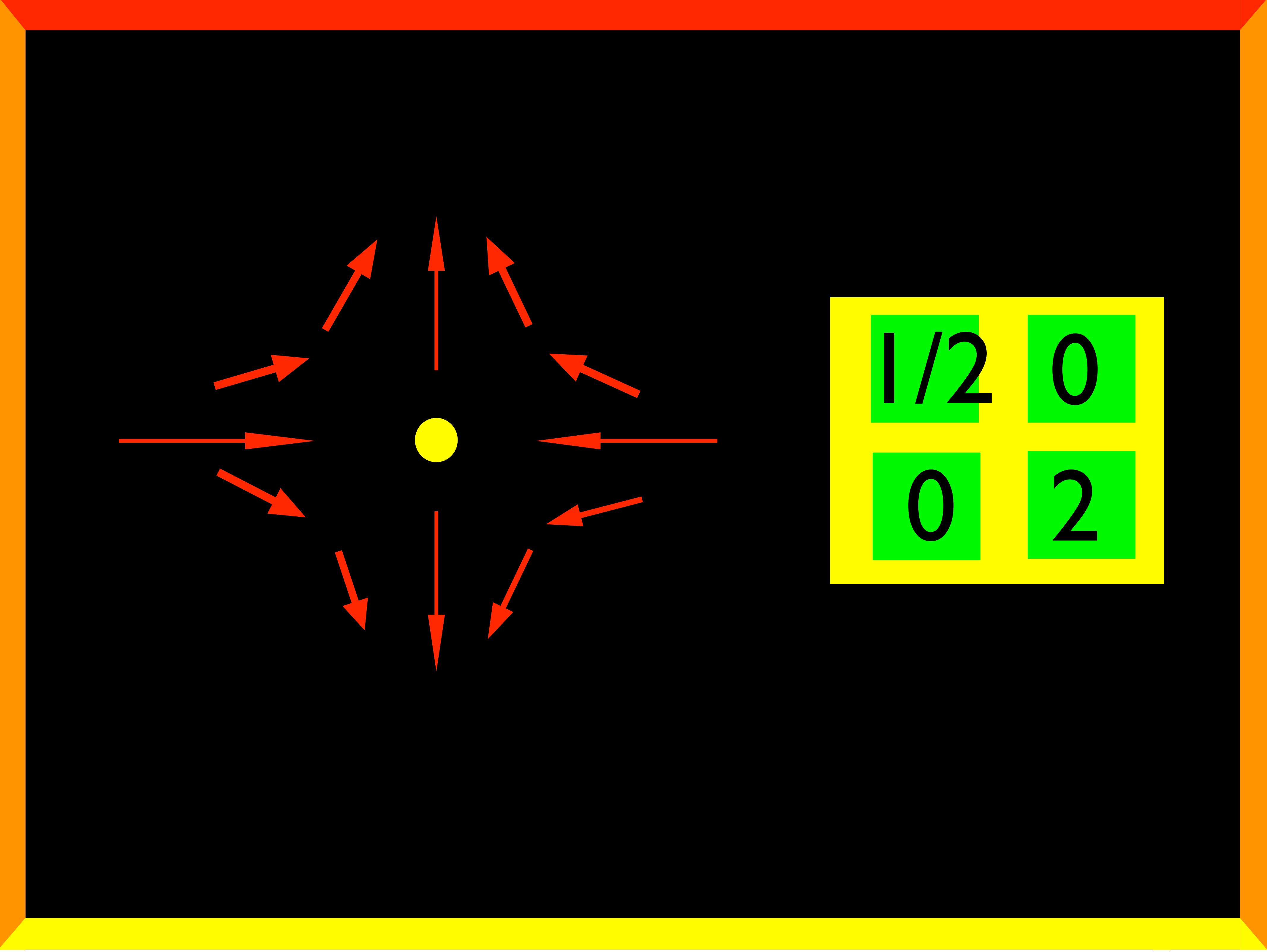
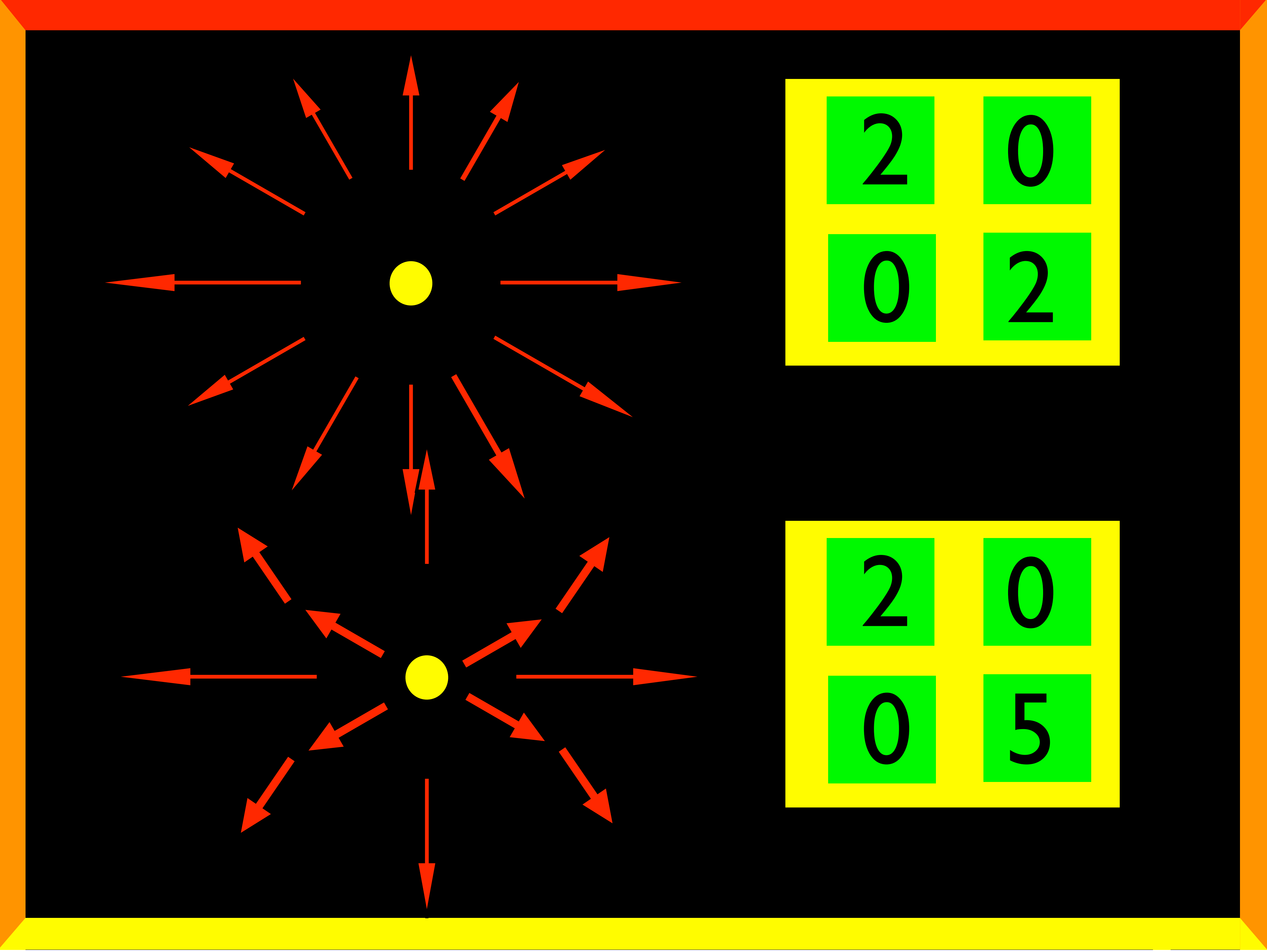
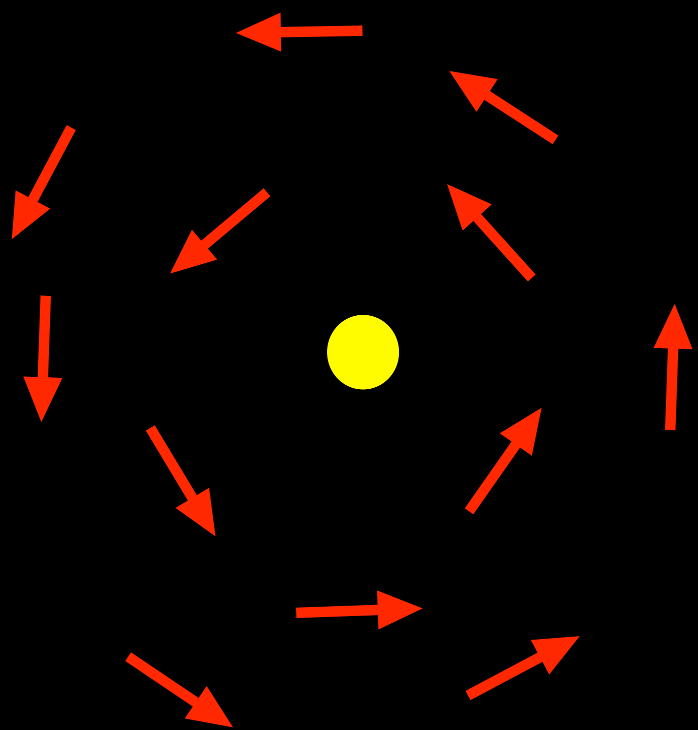
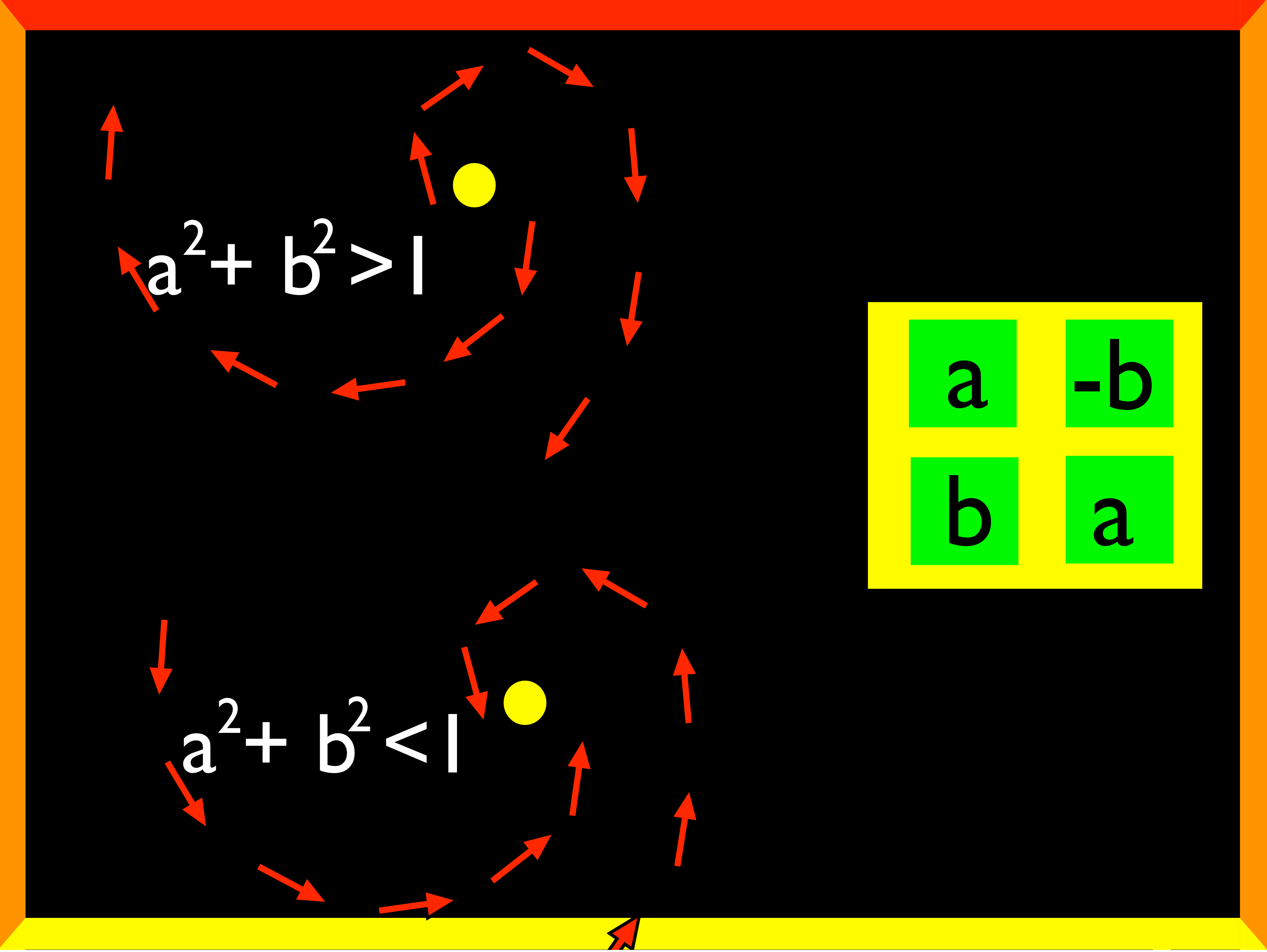

1/2 0
0 2


20
02
20
05

$$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$





$a^2 + b^2 > 1$
$a^2 + b^2 < 1$
a -b
b a

Discrete dynamical system case

Asymptotic stability:

All eigenvalues satisfy

$$|\lambda_k| < 1$$

PART II

Material since second midterm

We covered a lot since the second midterm!


Danny MacAskill

Before we continue:

Why is this course called 21b ??


LYDIA
51 63
5+1 6+3
6+9

A = 1

B = 2

21B = 21 + 2 = 23

Fibonnacci

1, 1, 2, 3, 5, 8, 13,...

Our final exam is on

May 18

which has only prime factors

2,3

Systems of linear differential equations

Continuous Systems

$$x' = Ax$$

$$x(0)=v$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

To solve this system $x' = A\,x$

$$A\,v_k = \lambda_k v_k$$

closed form solution

$$x(0) = c_1 v_1 + c_2 v_2$$

$$x(t) = e^{\lambda_1 t} c_1 v_1 + e^{\lambda_2 t} c_2 v_2$$

$x'(t) = 2x(t) + y(t)$ $\quad y'(t) = x(t) + 2y(t)$ $\quad x(0)=6, y(0)=2$

Here is half of what you have to know about differential equations

the mother of all differential equations:

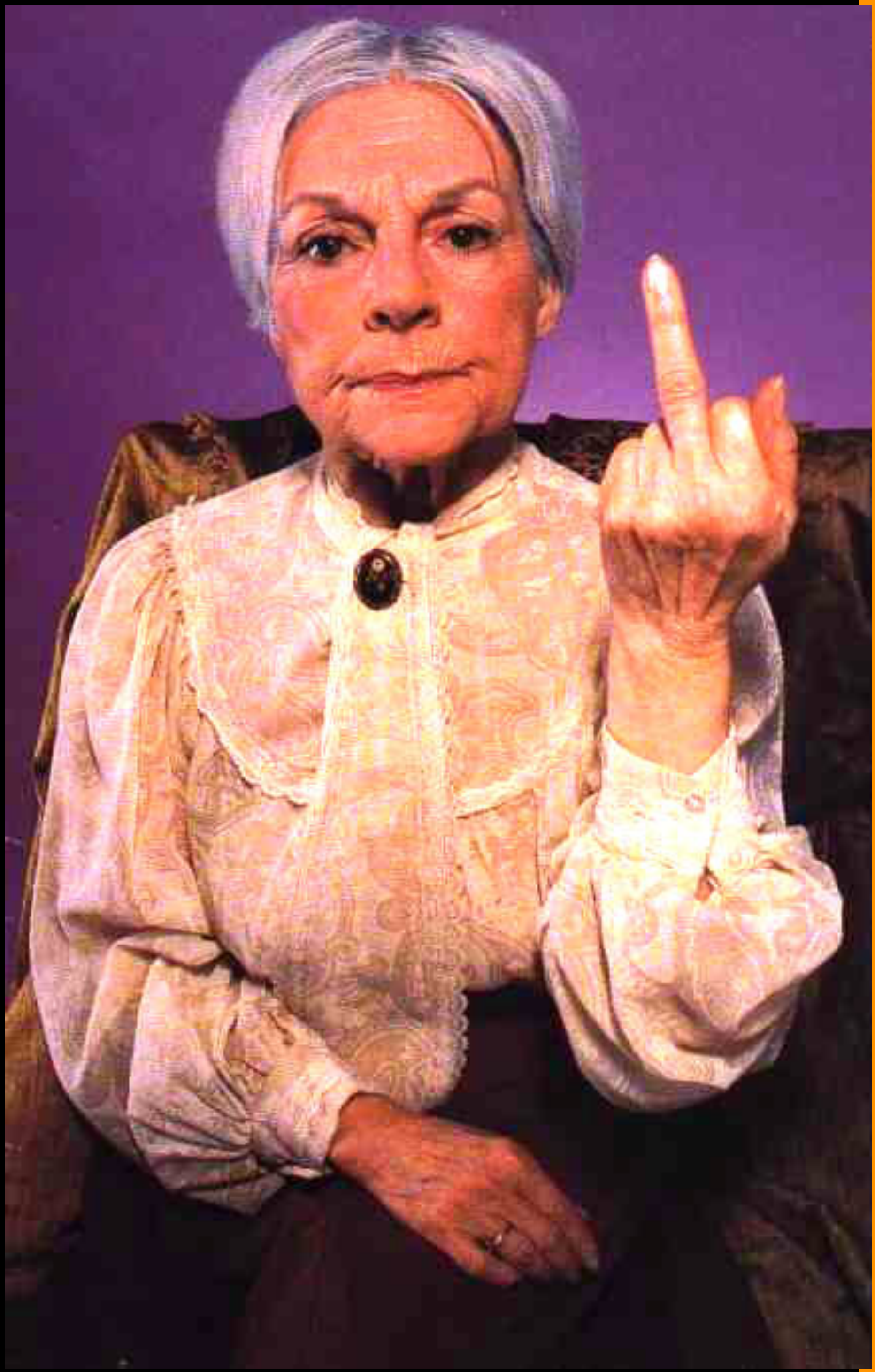
$$\frac{d}{dt}\, x(t) = \lambda\, x(t)$$

$$x(0) = b$$

It has the solution

$$x(t) = b\, e^{\lambda t}$$

If you should forget:



$$x' = \lambda x$$

Solution:

$$x(t) = x(0)\, e^{\lambda t}$$

THE MOTHER OF ODE'S

And here is the second half of what you have to know about differential equations x(t) = cos(kt) and x(t) = sin(kt) satisfy the differential equation

$$\ddot{x} = -k^2 x$$

The general solution is
A cos(k t) + B sin(k t)

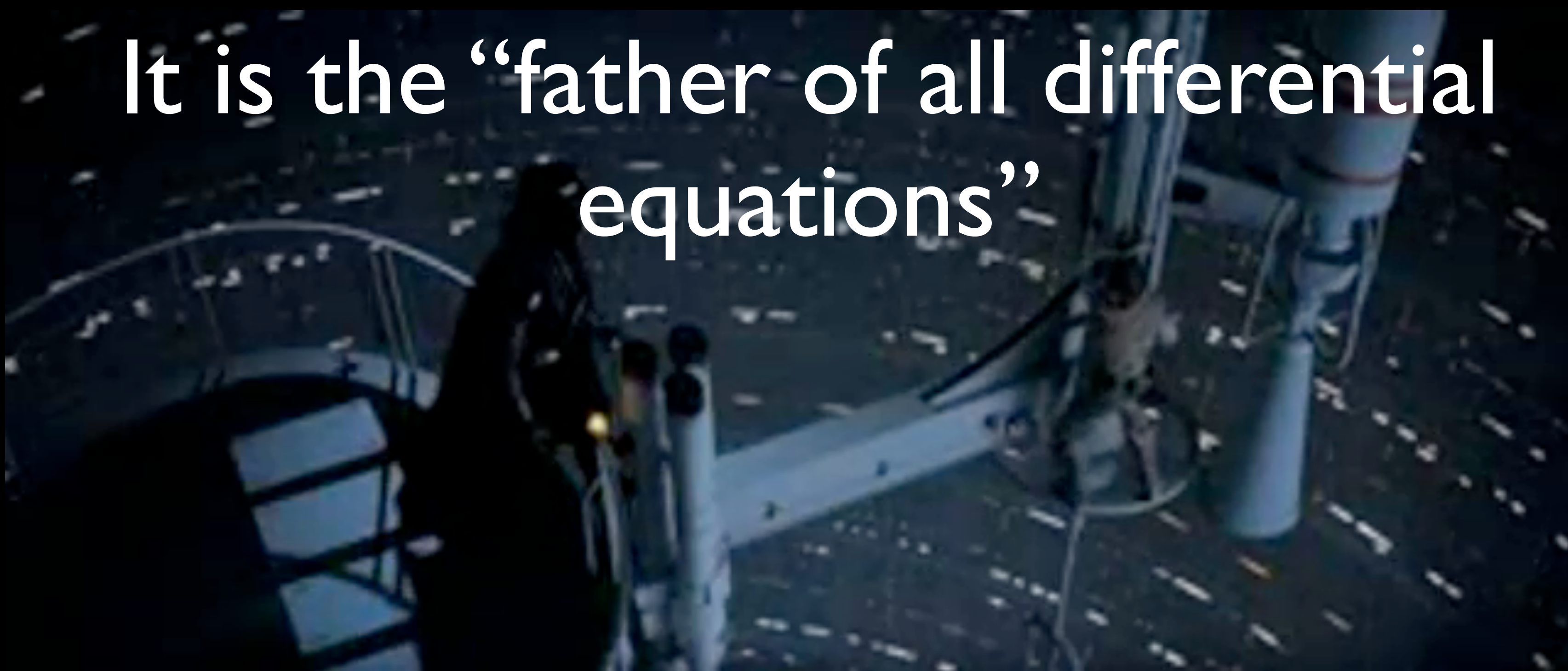
it appears in mechanics: using Newtons law

$$m\ddot{x} = -k^2 x$$

$$x(t) = A\cos(k\,t) + B\sin(k\,t)$$

$$A = x(0),\ \ B = x'(0)/k$$

This equation is called the harmonic oscillator
It is extremely important in physics.

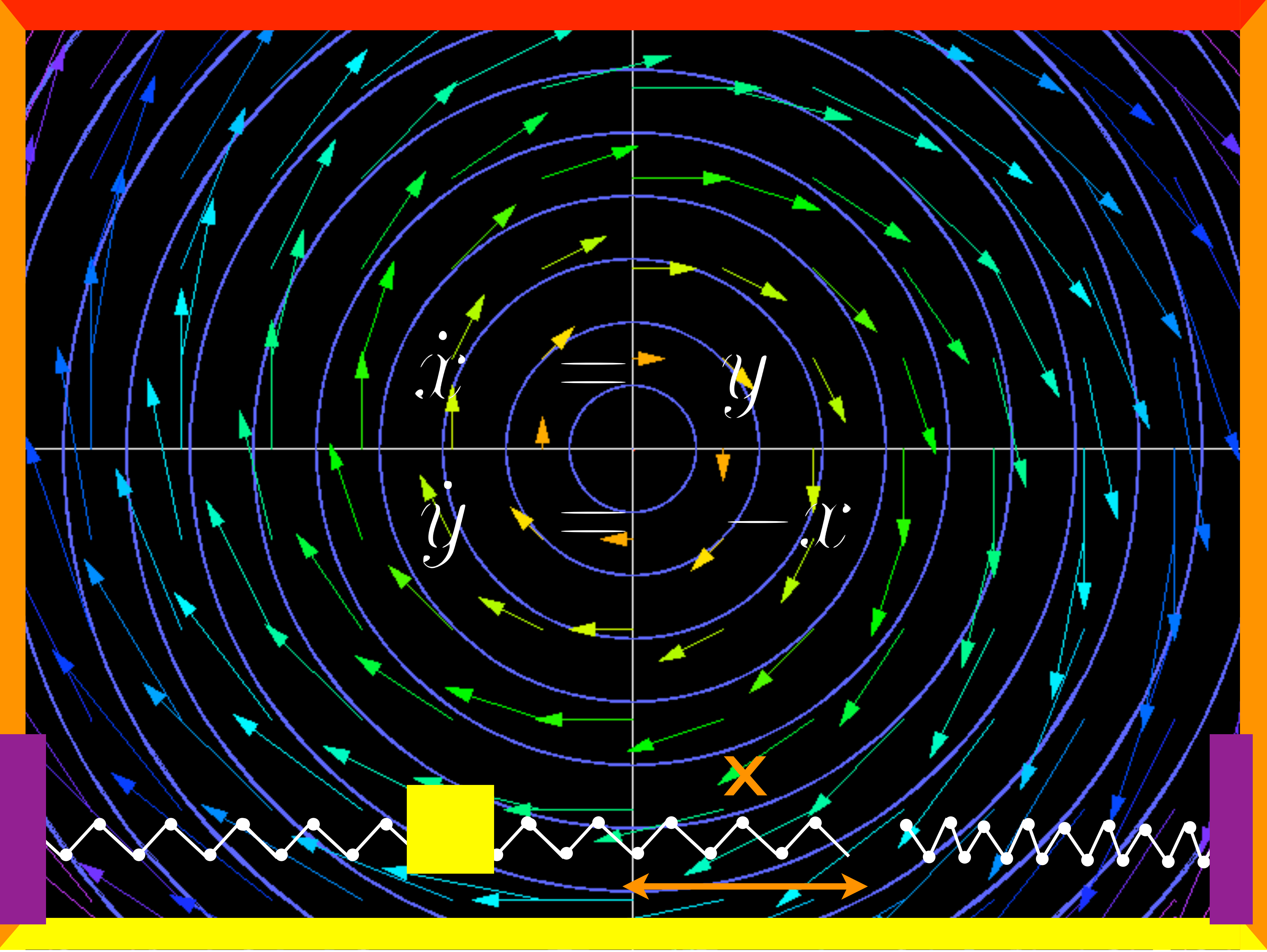


$$x'' = -c^2 x$$

Solution:

$$x(t) = x(0)\cos(c\,t) + x'(0)\sin(c\,t)/c$$

THE FATHER OF ODE'S


$\dot{x} = y$
$\dot{y} = -x$
x

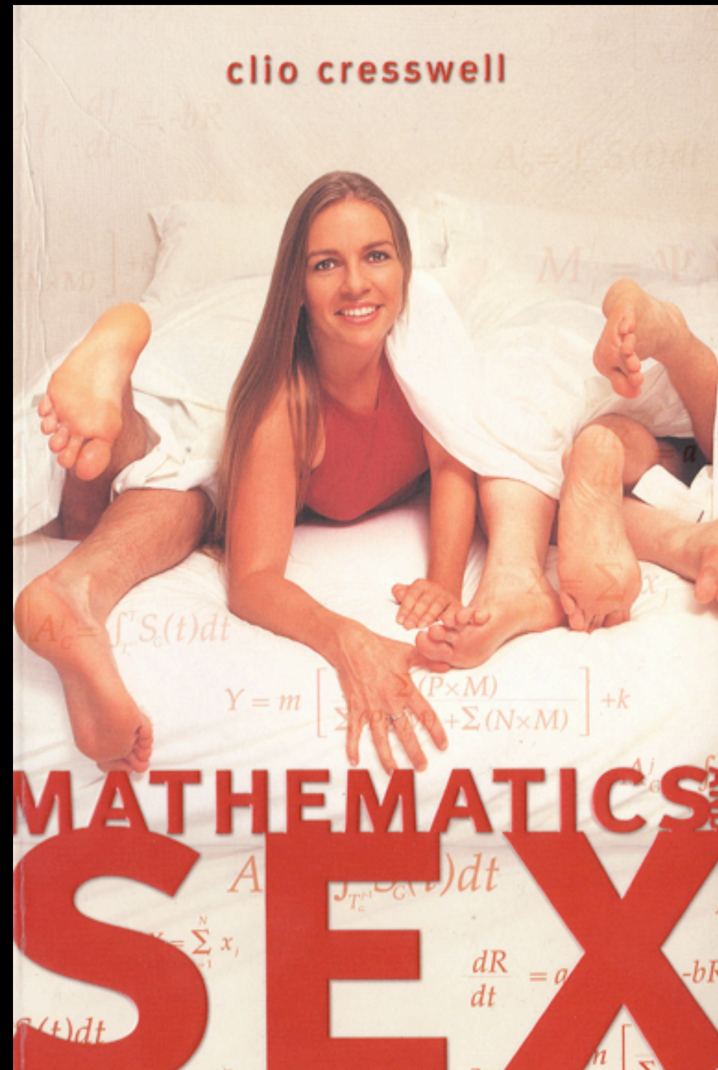
but it is also relevant in relationships

The harmonic oscillator appears also in a book of

Clio Cresswell

2003

Here is a scan of part of the book



LOVE, SWEEEET LOVE

In the late '80s, a Harvard lecturer by the name of Steven Strogatz suggested an unusual class exercise to his students. The day's topic would be the Mathematics of Love. Professor Strogatz's motivations were plain cheeky. Confronted with the challenge of capturing his students' attention on the predictive powers of equations, he reworded a common undergraduate mathematics problem into a language he thought the students would relate to: the evolution of the love affair between Romeo and Juliet. His ingenuity should not be taken lightly: turning a group of hormone-raging twentymake some mathematical sense of one of the great human emotions.

He presented the problem like this:

Romeo is in love with Juliet, but in our version of the story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

As you can see, emotions are a bit all over the place in this relationship. The question is, will they ever settle? What kind of relationship can Romeo and Juliet look forward to? The point

The first step towards mathematical insight is to rewrite the terms of Romeo and Juliet's fickle affair mathematically. The translation is:

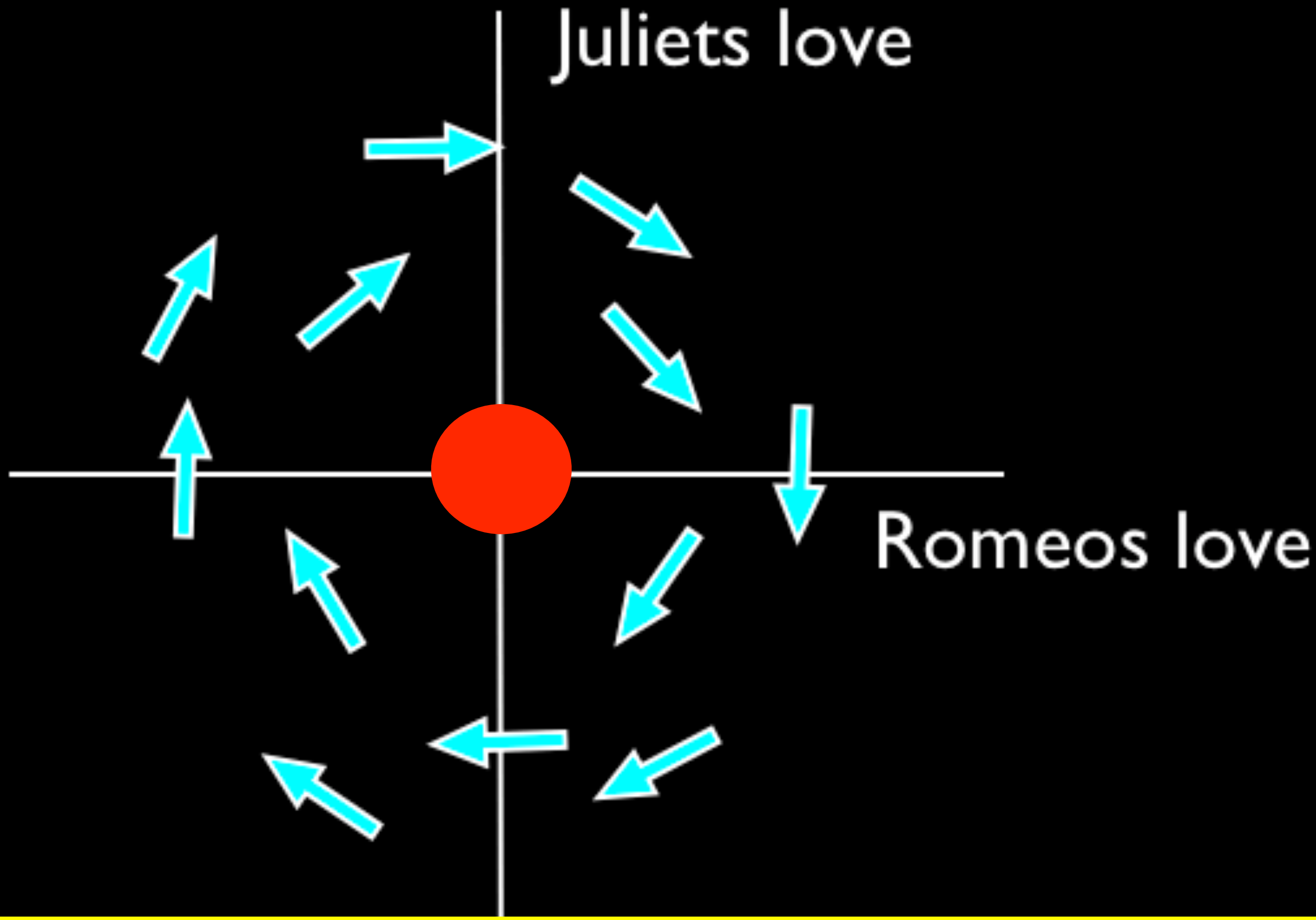
$$\frac{dR}{dt} = aJ, \frac{dJ}{dt} = -bR,$$

where R is for Romeo, and J for Juliet. How the letters are combined mimics how Romeo and Juliet find themselves interacting. For mathematicians, translating the problem into equations like this is natural. Mathematics is the study of patterns and this problem simply concerns behavioural patterns. Behavioural patterns are not static though and that's an important characteristic to

Romeo and Juliet

Romeo warms up when given more love, Juliet wants to run away when being desired.

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$



if we add a damping factor:

$$\ddot{x} = -b\dot{x} - k^2 x$$

we have a situation relevant in love also as tabloids show us. Lets look at this later.

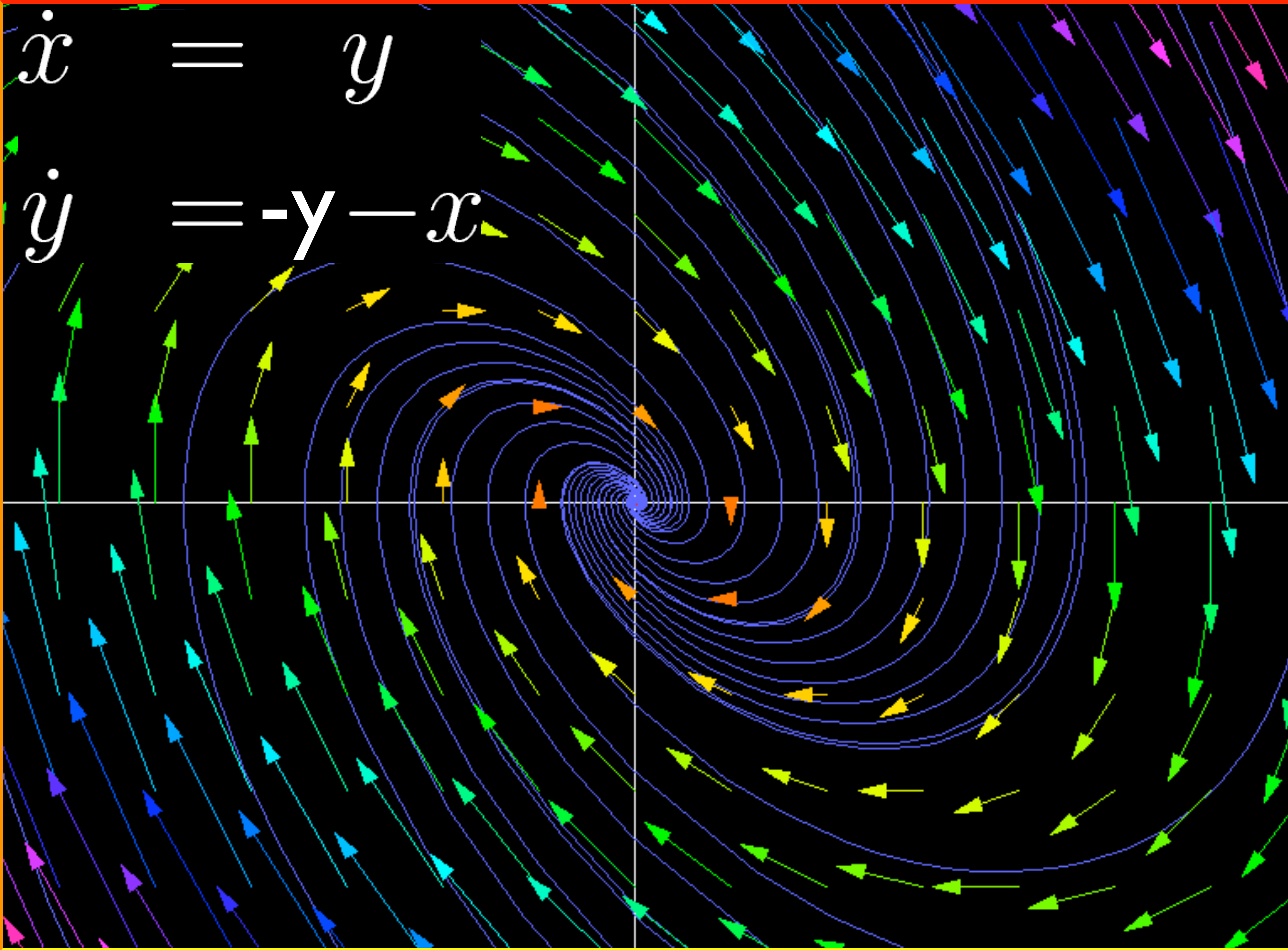
it is good to know that it has the solution

$$x(t) = e^{-bt} \left(A \cos(c\,t) + B \sin(c\,t) \right)$$

where $-b + i\,c$ are the roots of the characteristic polynomial.

$$\dot{x} = y$$

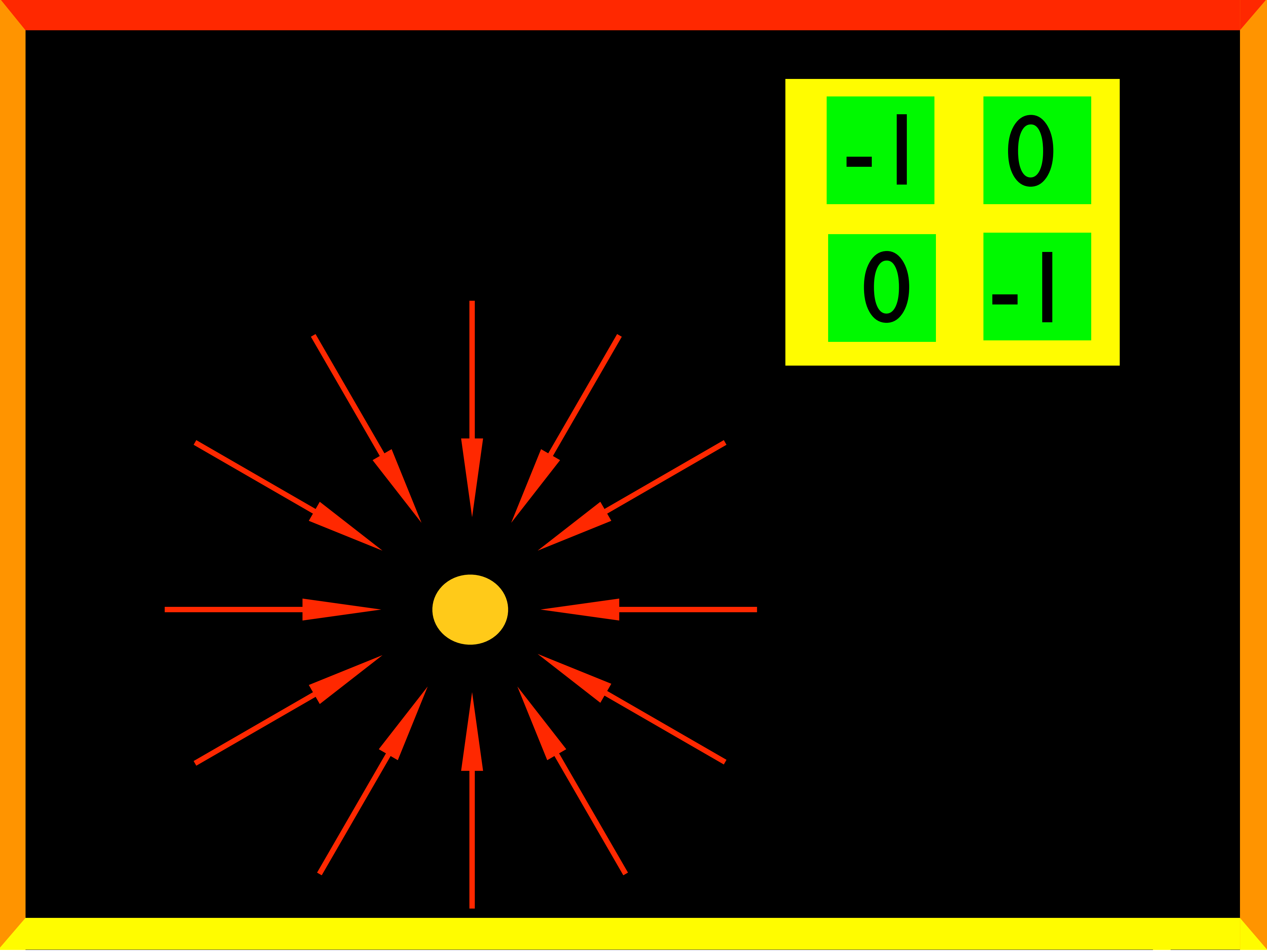
$$\dot{y} = \text{-y} - x$$

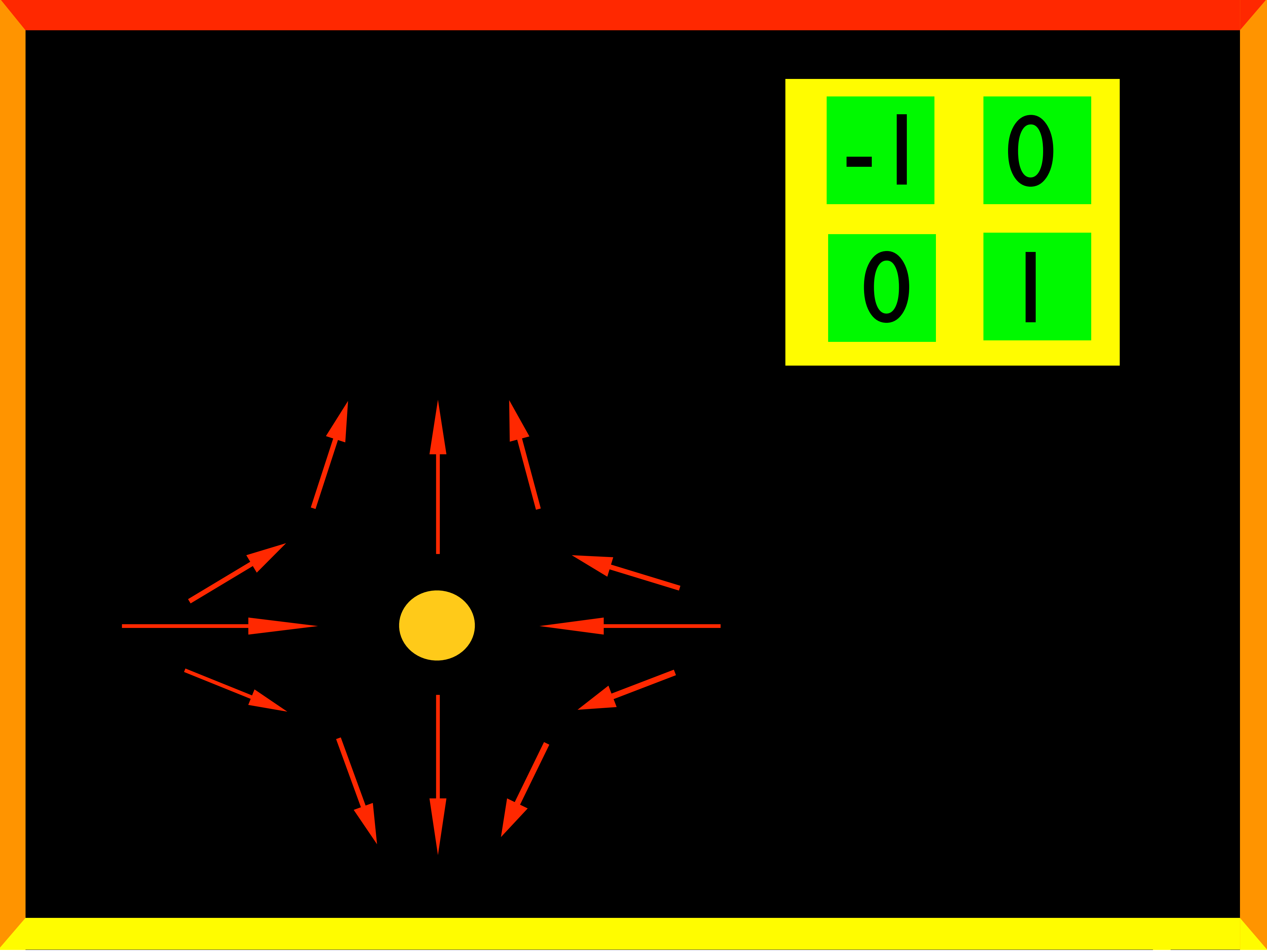


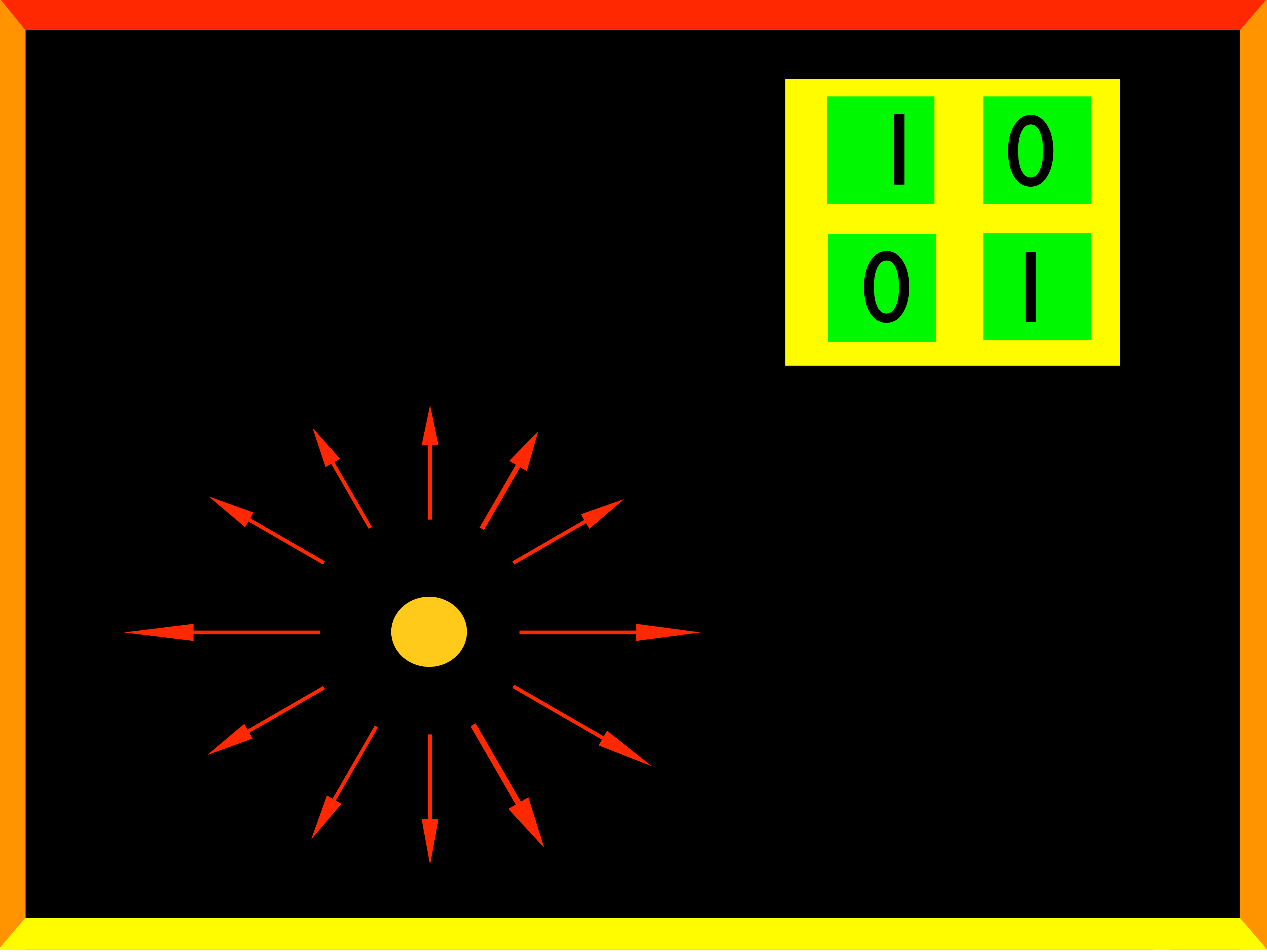
Stability of continuous dynamical systems

The real part of the eigenvalues is relevant!

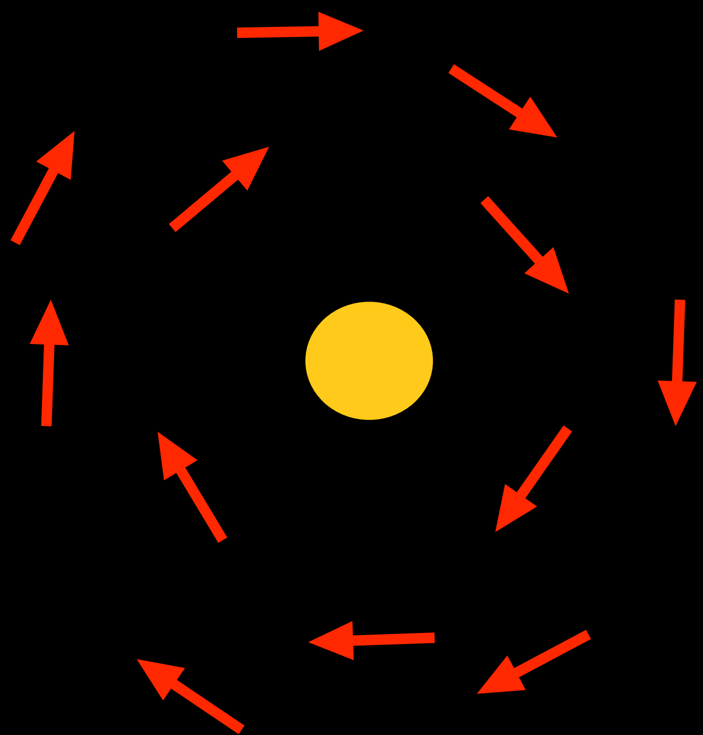
Proof: ask Mother!


-1 0
0 -1

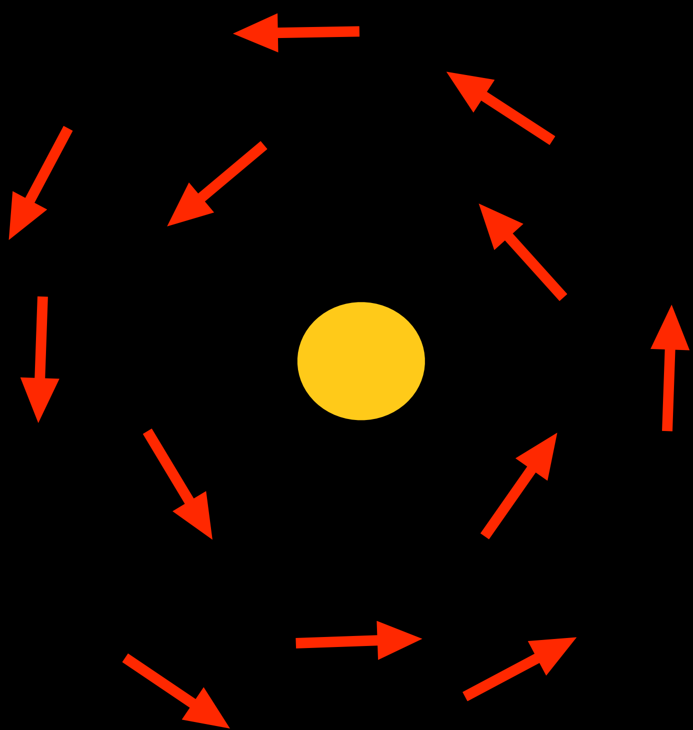

-1 0
0 1


I
0
0
I

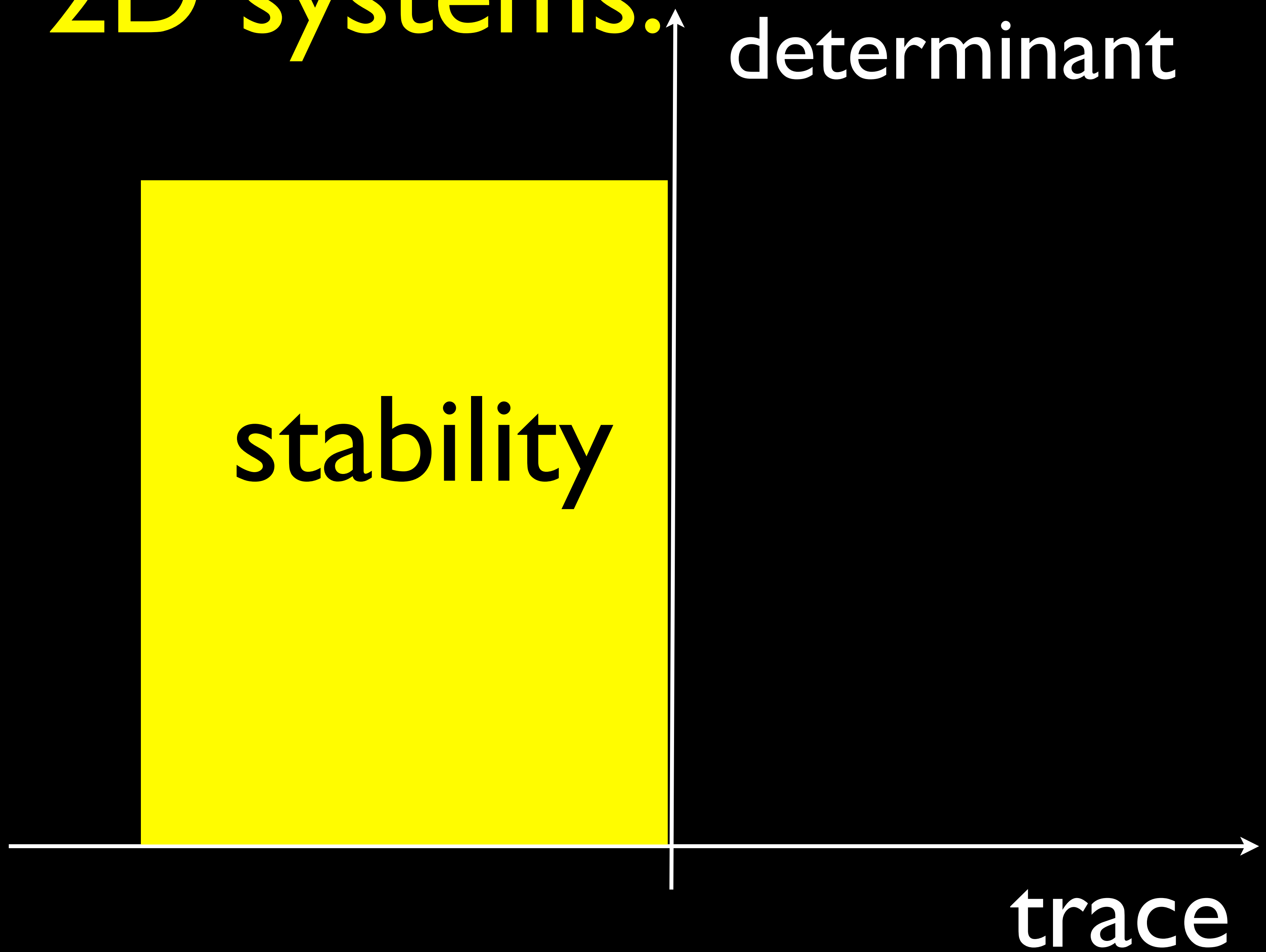
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



for 2D systems:

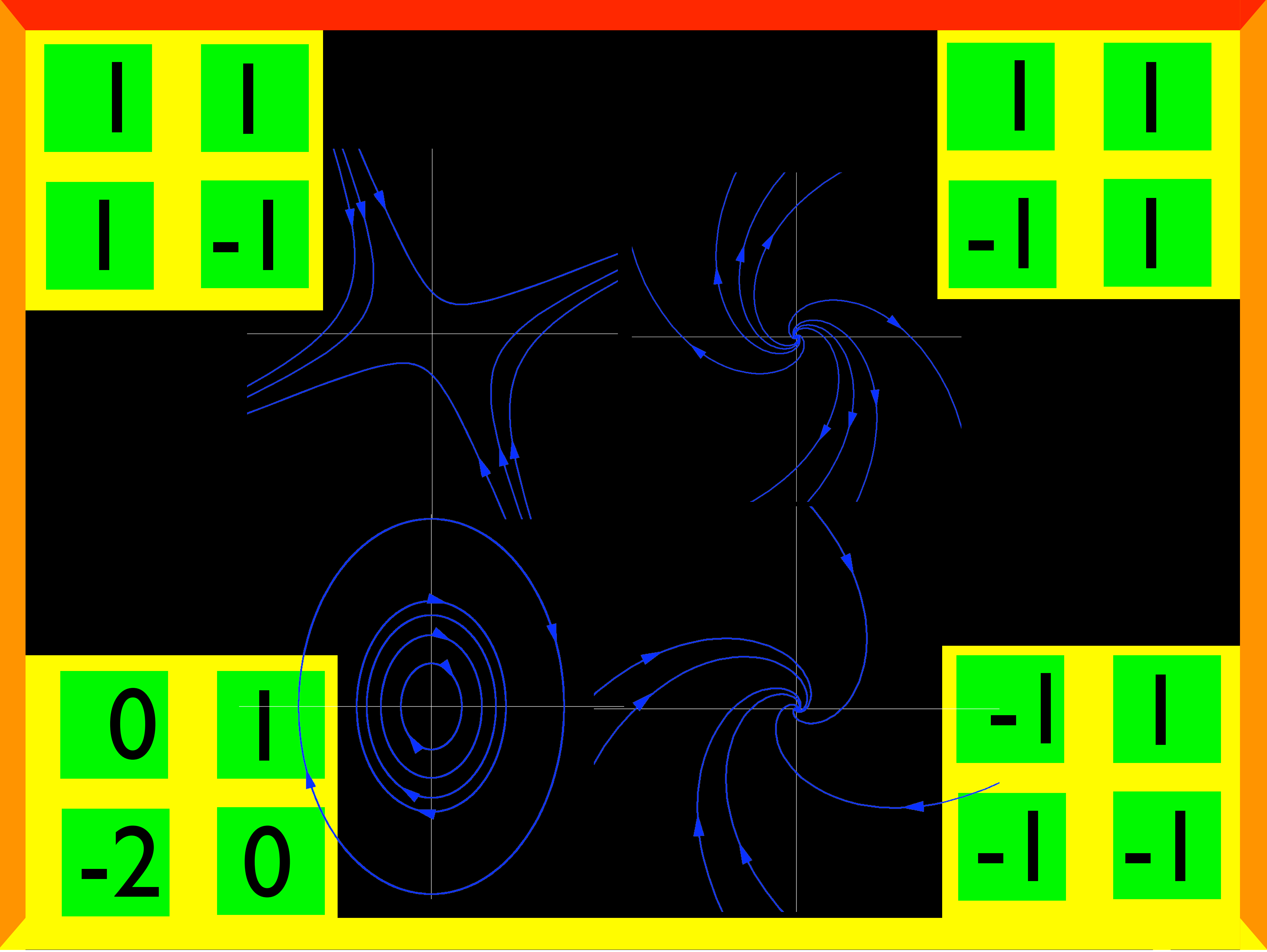


problem

Determine the stability of the systems with the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$



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Inhomogeneous differential equations

There are two different ways to solve it:

operator method

cookbook method

Differential operators

$$D f = f'$$

$p(D)$ polynomial

$$p(D)=(D-\lambda_1)\ldots(D-\lambda_n)$$

fundamental theorem of algebra

To solve a inhomogeneous differential equation

$$p(D)\, f = g$$

factor p(D) into linear factors and invert each linear factor

$$f=(D-\lambda_1)^{-1}\ldots(D-\lambda_n)^{-1}g$$

using the inversion formula for linear differential operators

$$(D-\lambda)^{-1}g = ce^{\lambda x} + e^{\lambda x}\int_0^x e^{-\lambda t}g(t)\,dt$$

Allows to solve all problems without exception, but needs integration skills.

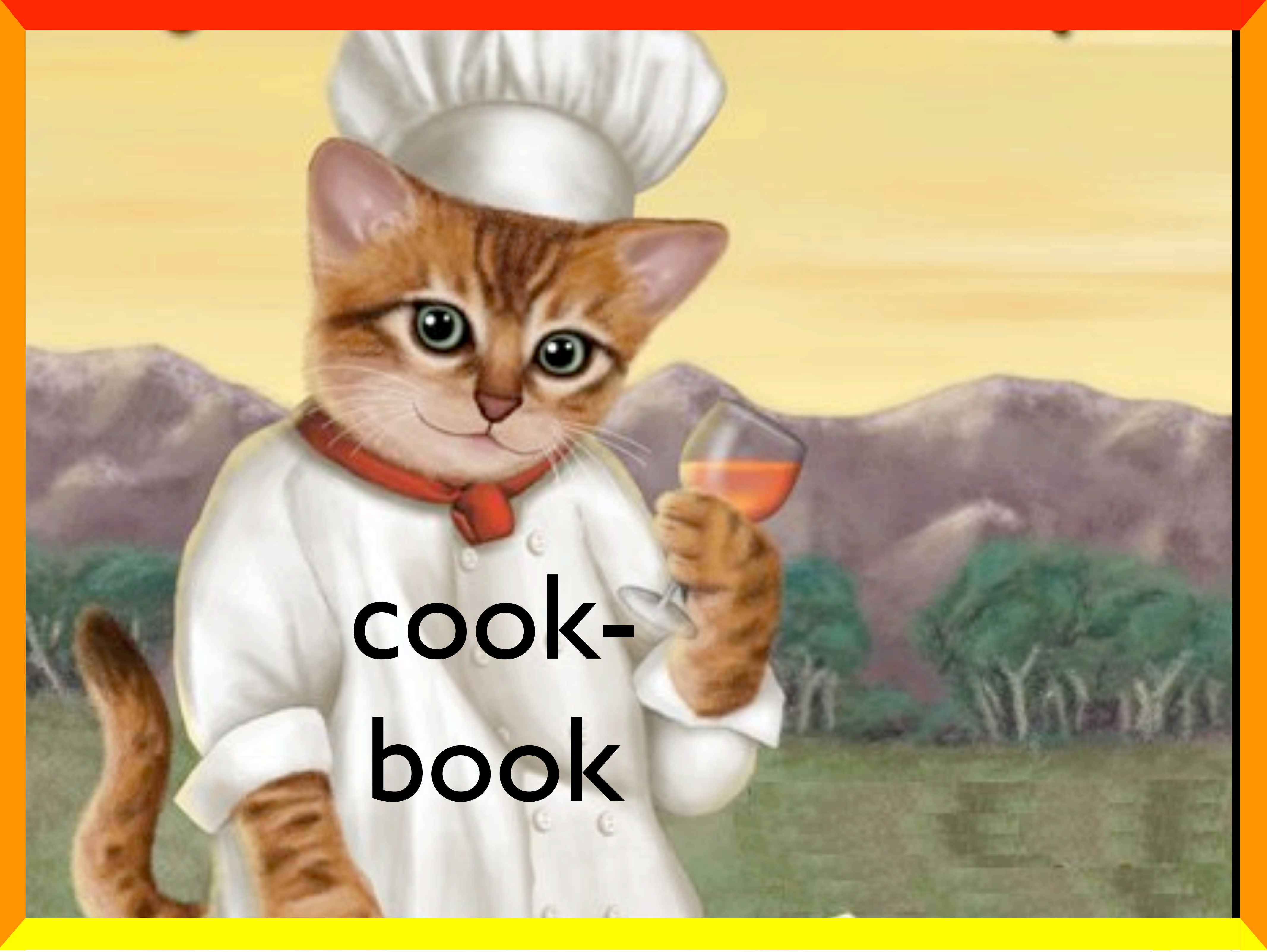
Example with operator method

$$f'' + 10 f' + 21 f = 1$$

$$(D+3)(D+7) f = 1$$

$$(D+3)^{-1} t = A e^{-3t} + 1/3$$

$$(D+7)^{-1} [A e^{-3t} + 1/3] = A e^{-3t} + B e^{-3t} + 1/21$$


cook-
book

Cookbook

Solution of p(D) f = g

Solve the homogeneous problem
p(D) f = 0

Find a special solution

For second order equations, there are three cases

- different real roots
- same real roots
- complex conjugated roots

problem

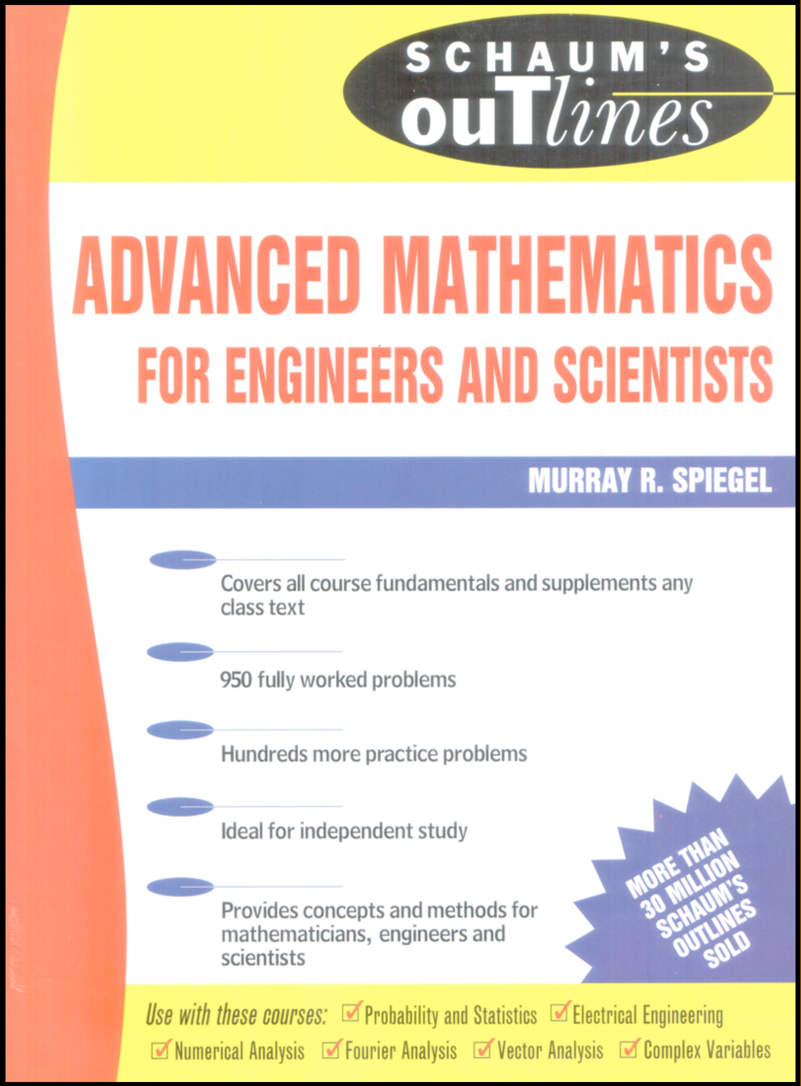
Example 1 $f'' + 9f = 2\sin(2t)$

Example 2: $f'' + f = e^{2t}$

Example 3: $f'' - 3f' + 2f = e^{2t}$

Example 4: $f'' - 4f' + 4f = e^{2t}$

The cookbook method is popular among engineers and is often faster than the operator method



How do we guess the special solution?

right hand side	Try with
$\sin(kt)$	$A \sin(kt) + B \cos(kt)$
$e^{k t}$	$e^{k t}$
1	c
t	$at+b$
t^2	$at^2+bt + c$
if already homogenous solution, take	$te^{k t}$ $At \sin(kt) + B\, t \cos(kt)$

problem

Pick an other example

$$f' - 3f = e^t$$

$$f'' - 6f' + 9f = e^t$$

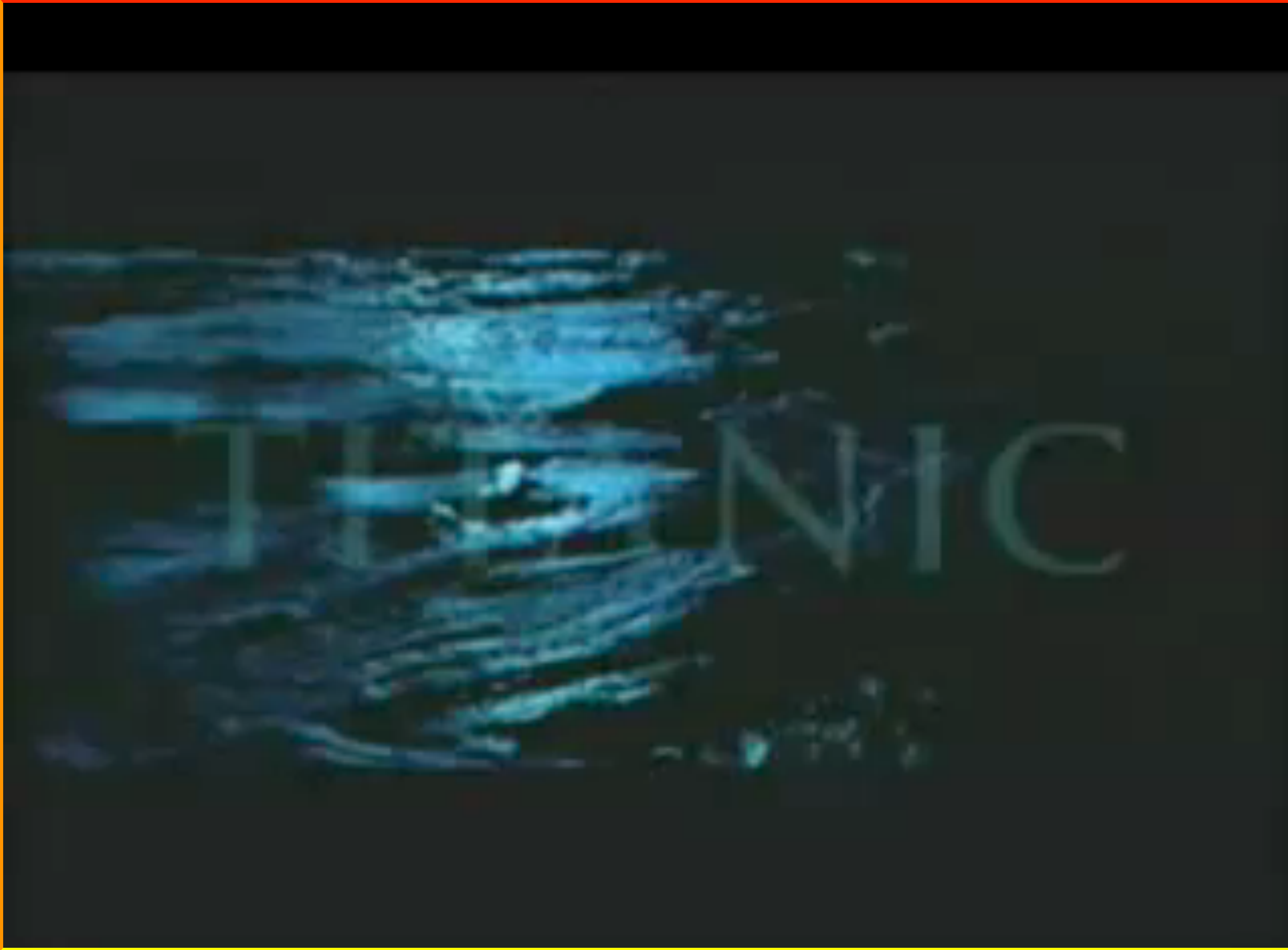
$$f'' + 9f = e^t$$

$$f'' + 2f' + f = t$$

$f' - 3f = e^t$ $f'' - 6f' + 9f = e^t$ $f'' + 2f' + f = t$

$f'' + 9f = e^t$

Before we have a quiz
with a DVD to win,
wouldn’t it be nice
to summarize the
entire course
in 5 seconds?



THE
LION KING

2lb in 5 Seconds

DVD to win.

The first correct solution

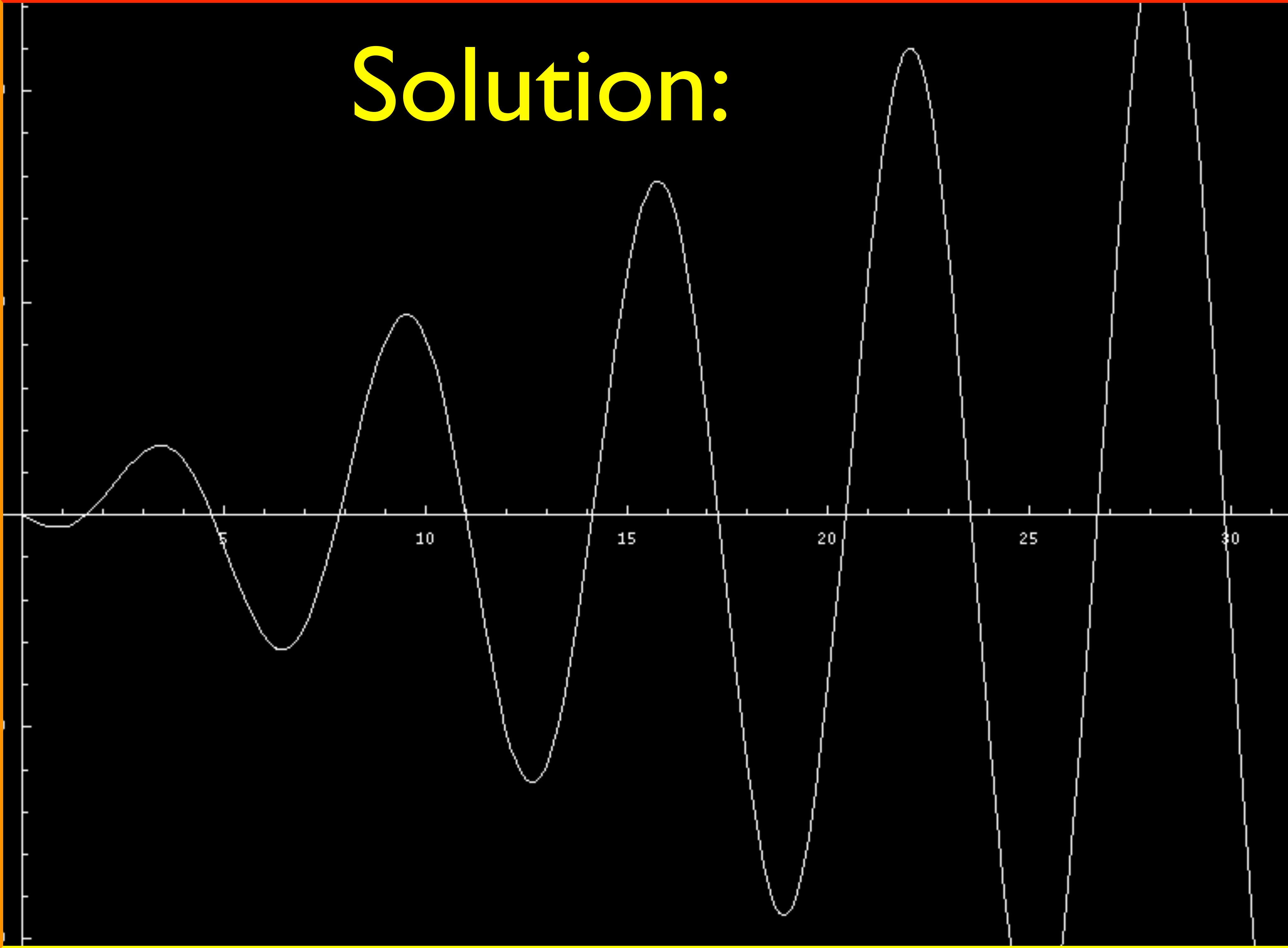
Find a special solution of

$$f''(t) - f'(t) = 17$$

An last example:

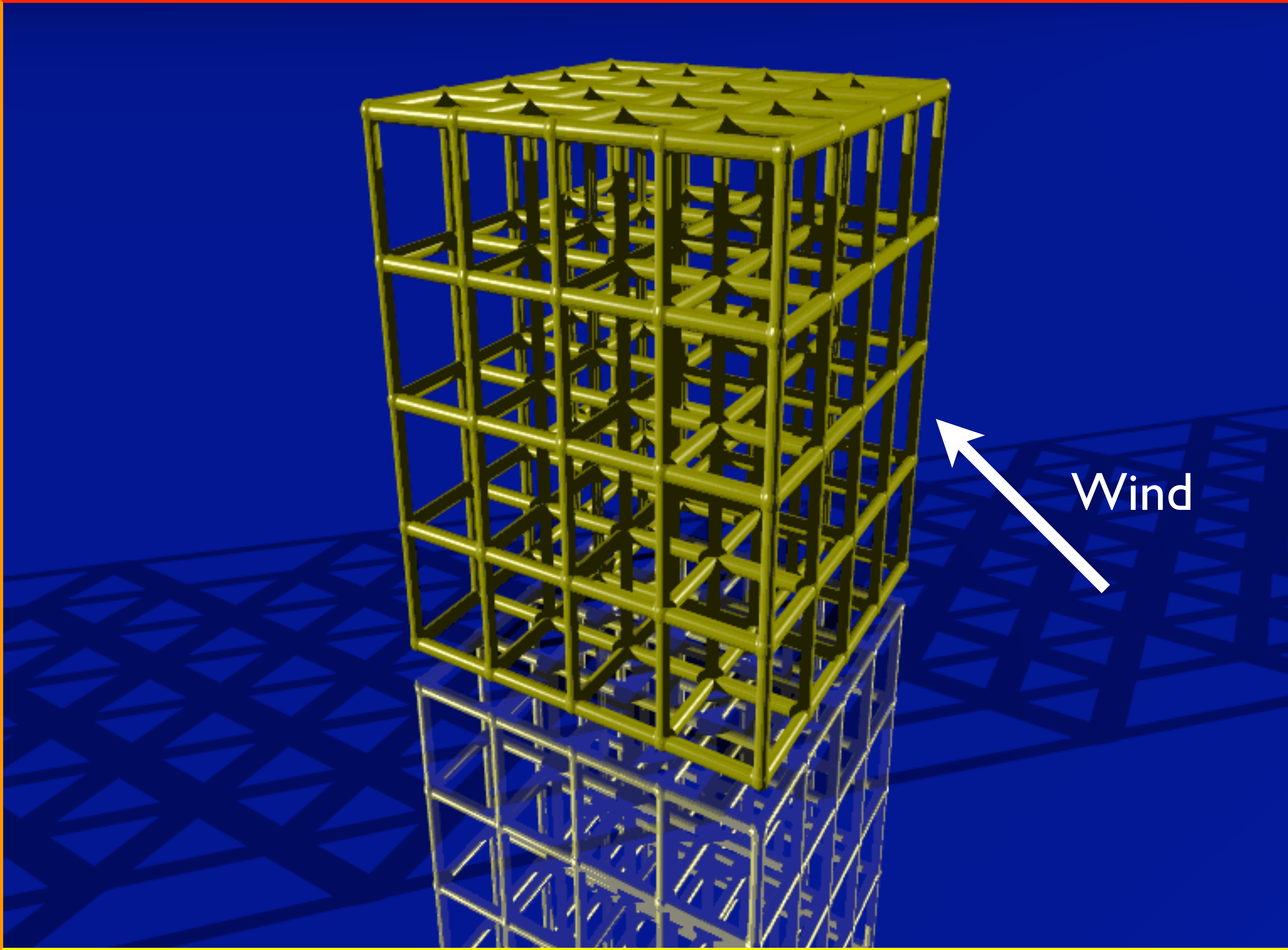
$$f''(t) + f(t) = \sin(t)$$

Solution:



Apropos resonance: the Hancock Tower

- 790 feet=241 m
- 60 story
- 300 ton mass dampeners
- constructed 1973-1976
- 45th tallest US building
- 131 tallest world
- 16 Herz motion


Wind

Nonlinear systems

We look at equations in the plane

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

An example

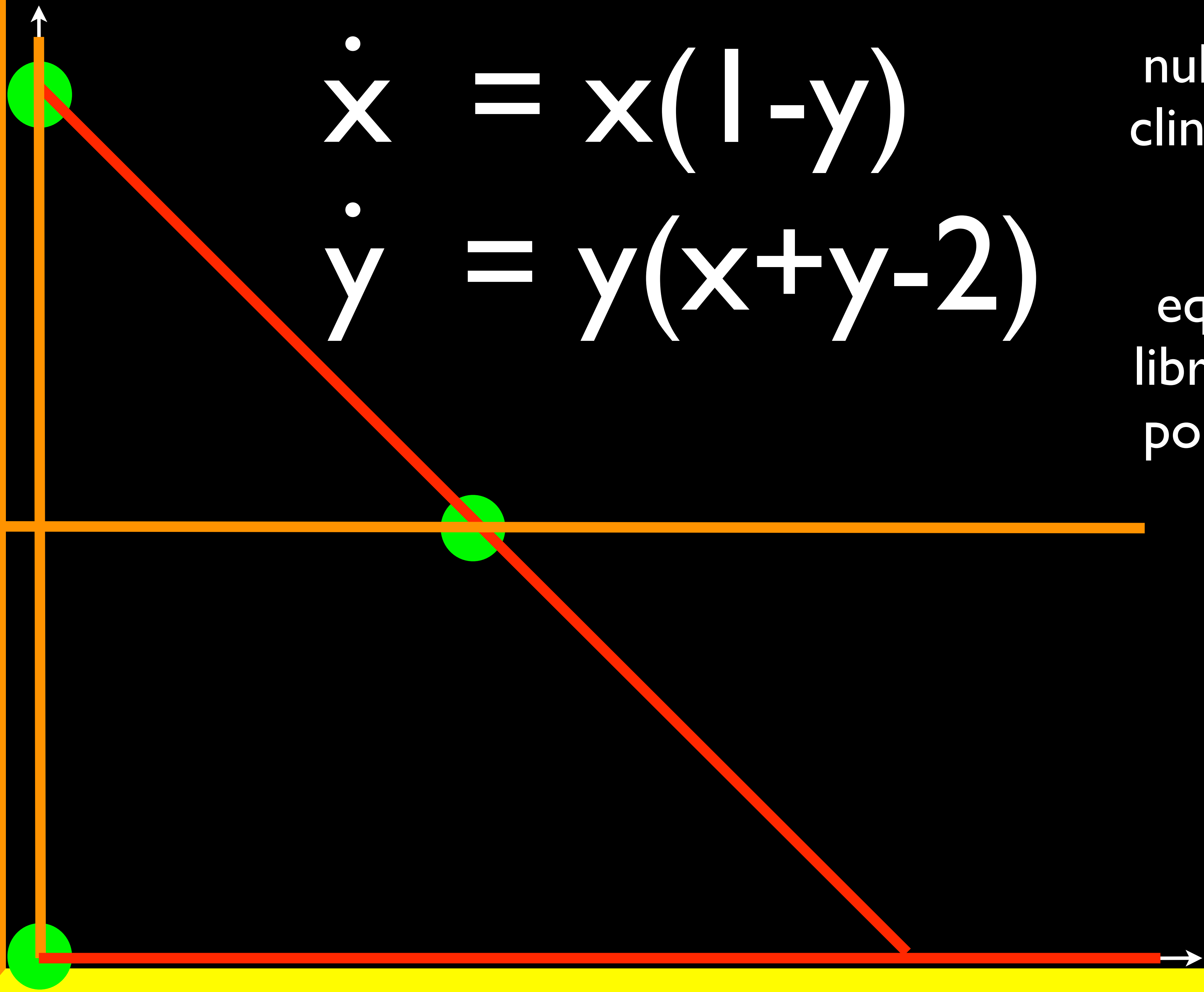
$$\dot{x} = x(1-y)$$
$$\dot{y} = y(x+y-2)$$

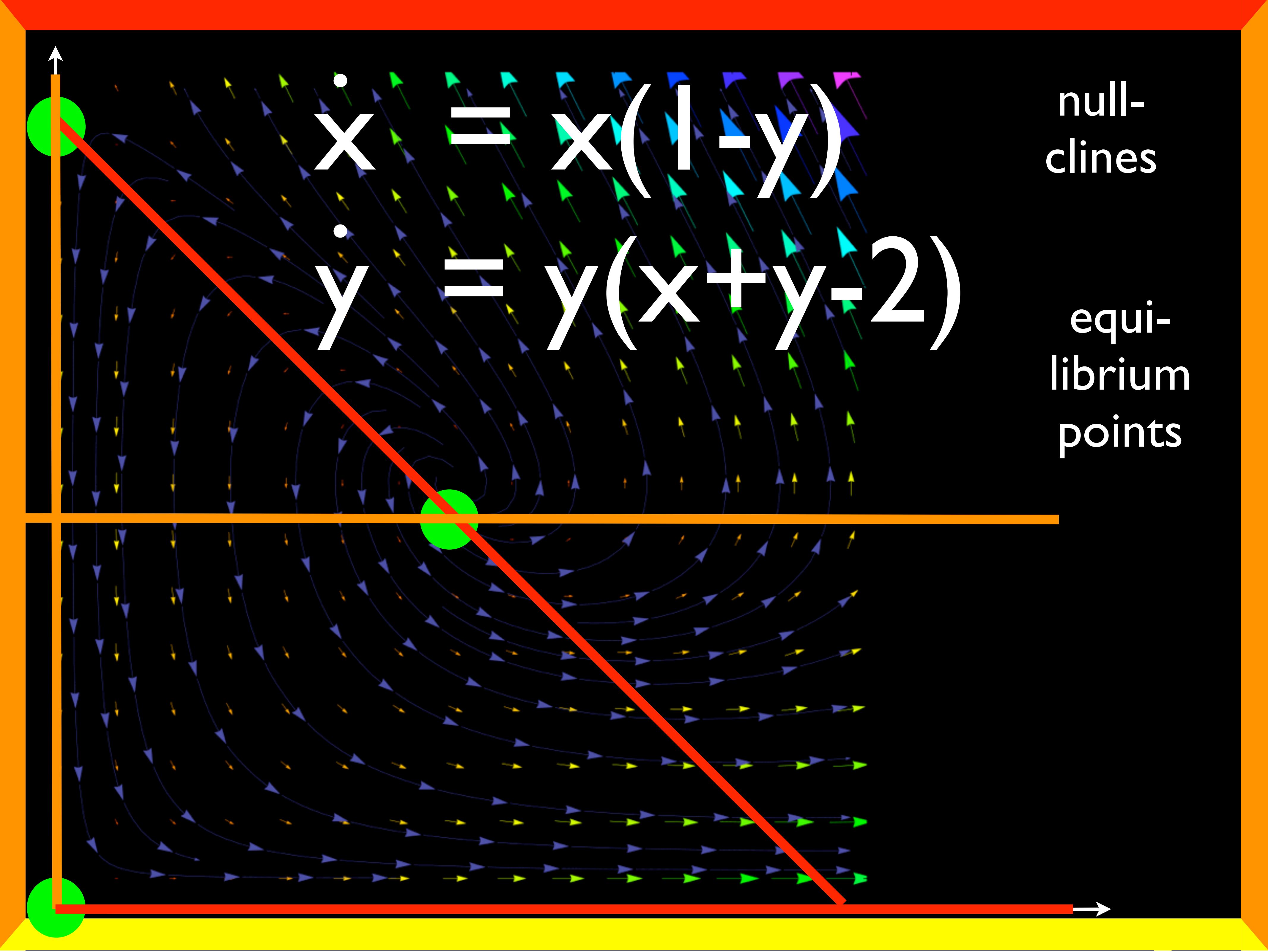
$$\dot{x} = x(1-y)$$

$$\dot{y} = y(x+y-2)$$

null-
clines

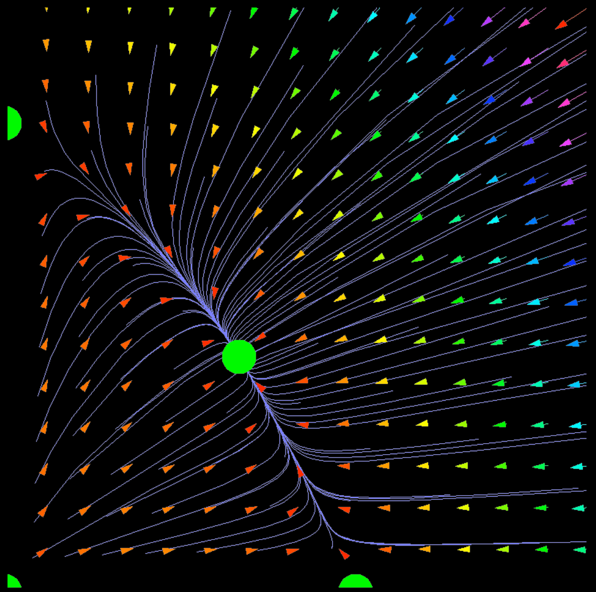
equi-
librium
points





$\dot{x} = x(1-y)$
$\dot{y} = y(x+y-2)$
null-clines
equi-librium points

Jacobean matrix

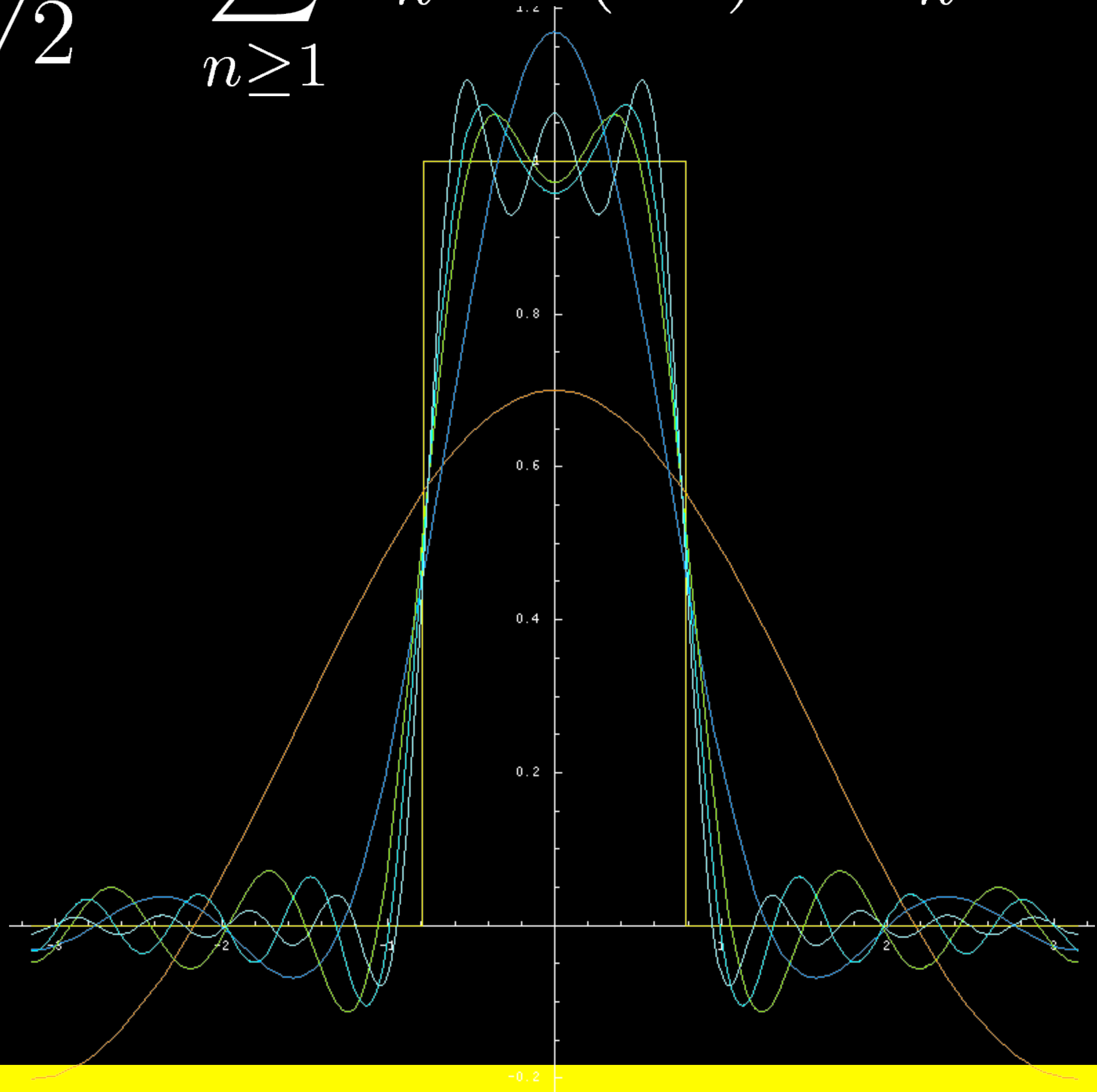


$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Fourier analysis

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n\geq 1} a_n \cos(nx) + b_n \sin(nx)$$



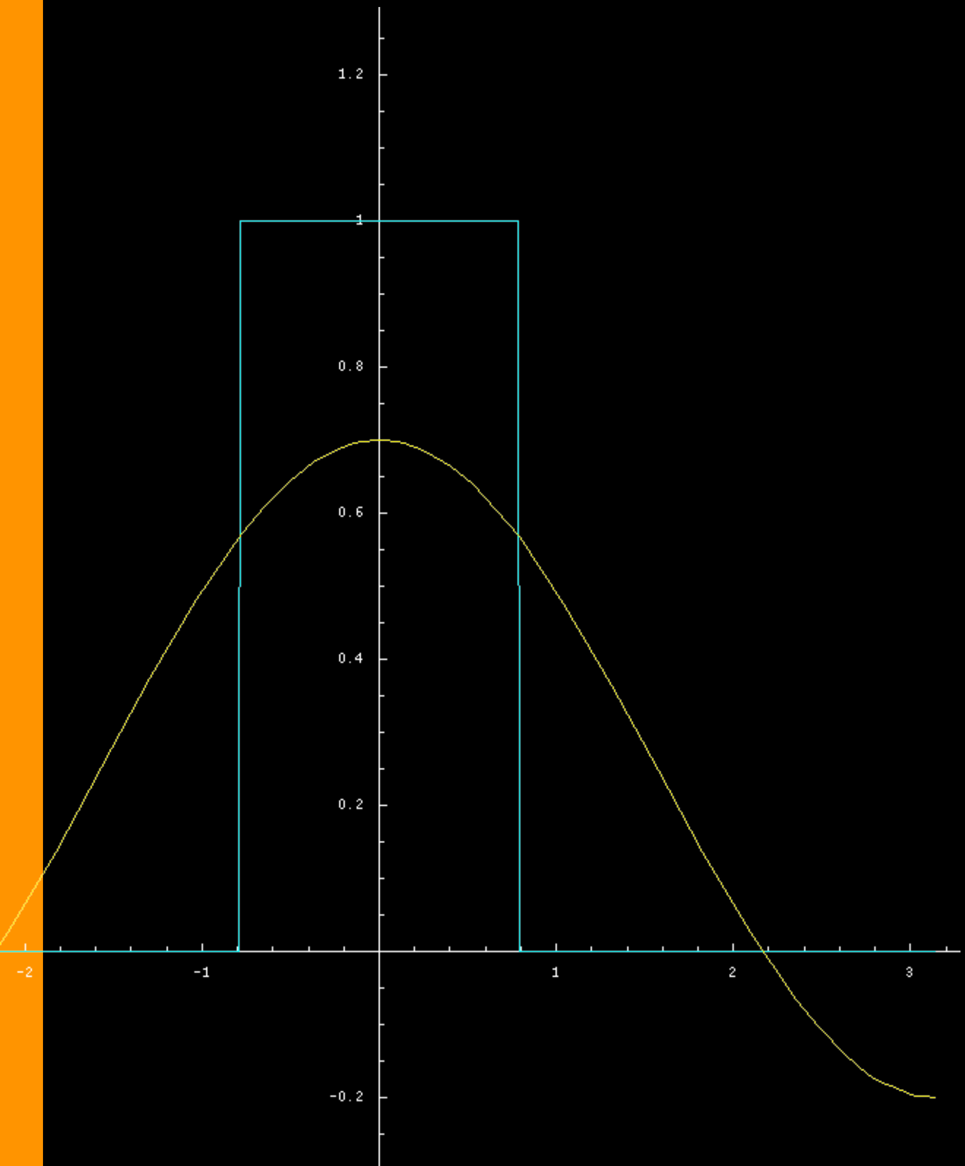
Fourier coefficients:

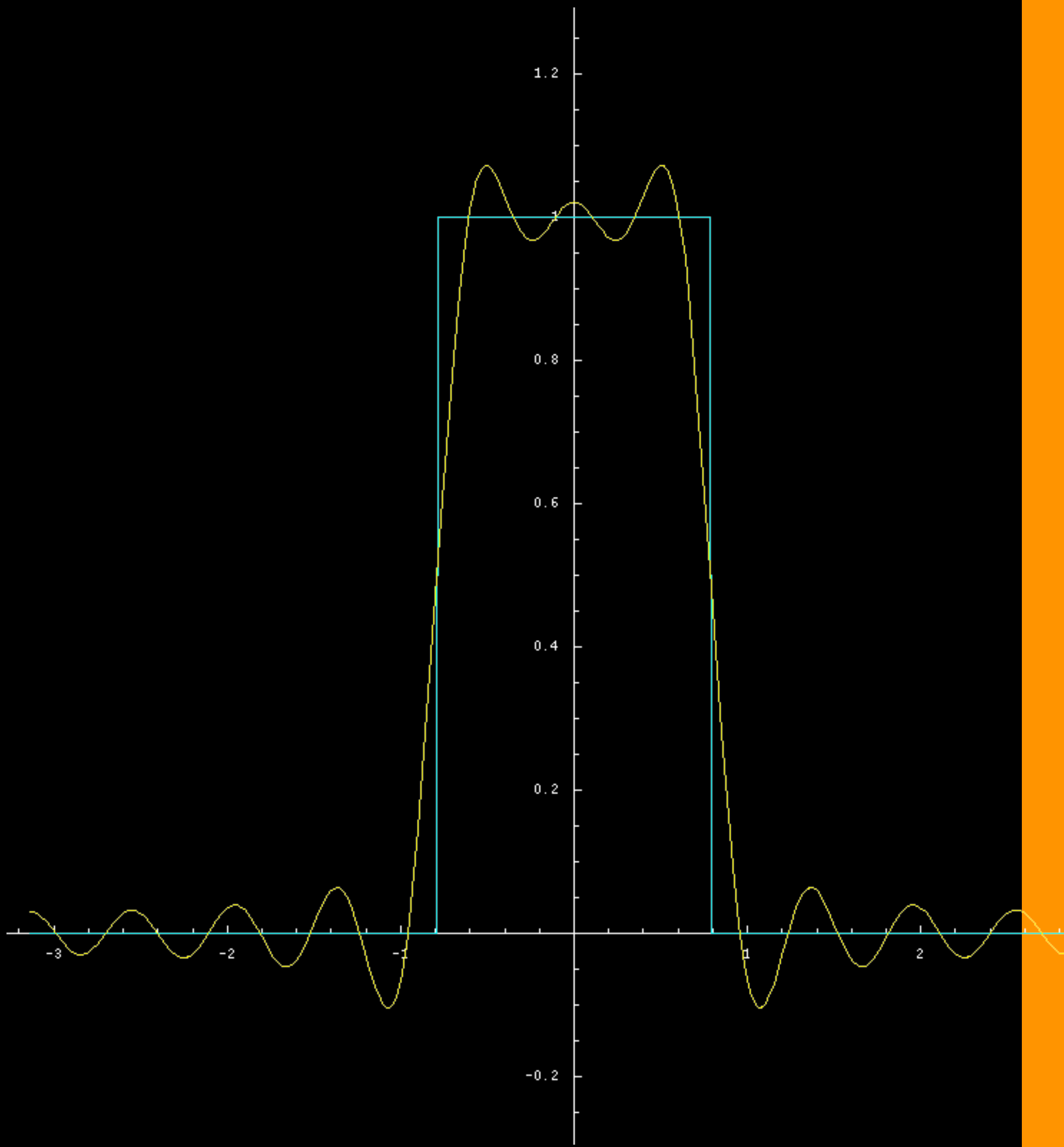
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2}} \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \; dx$$

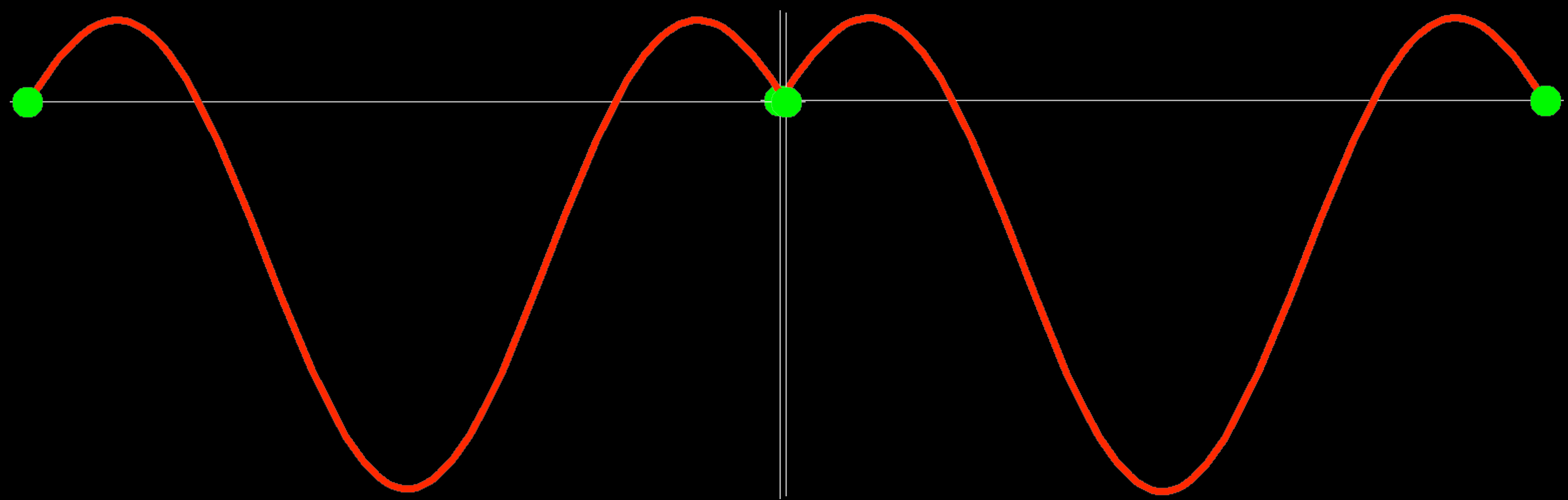
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \; dx$$

Fourier approximation



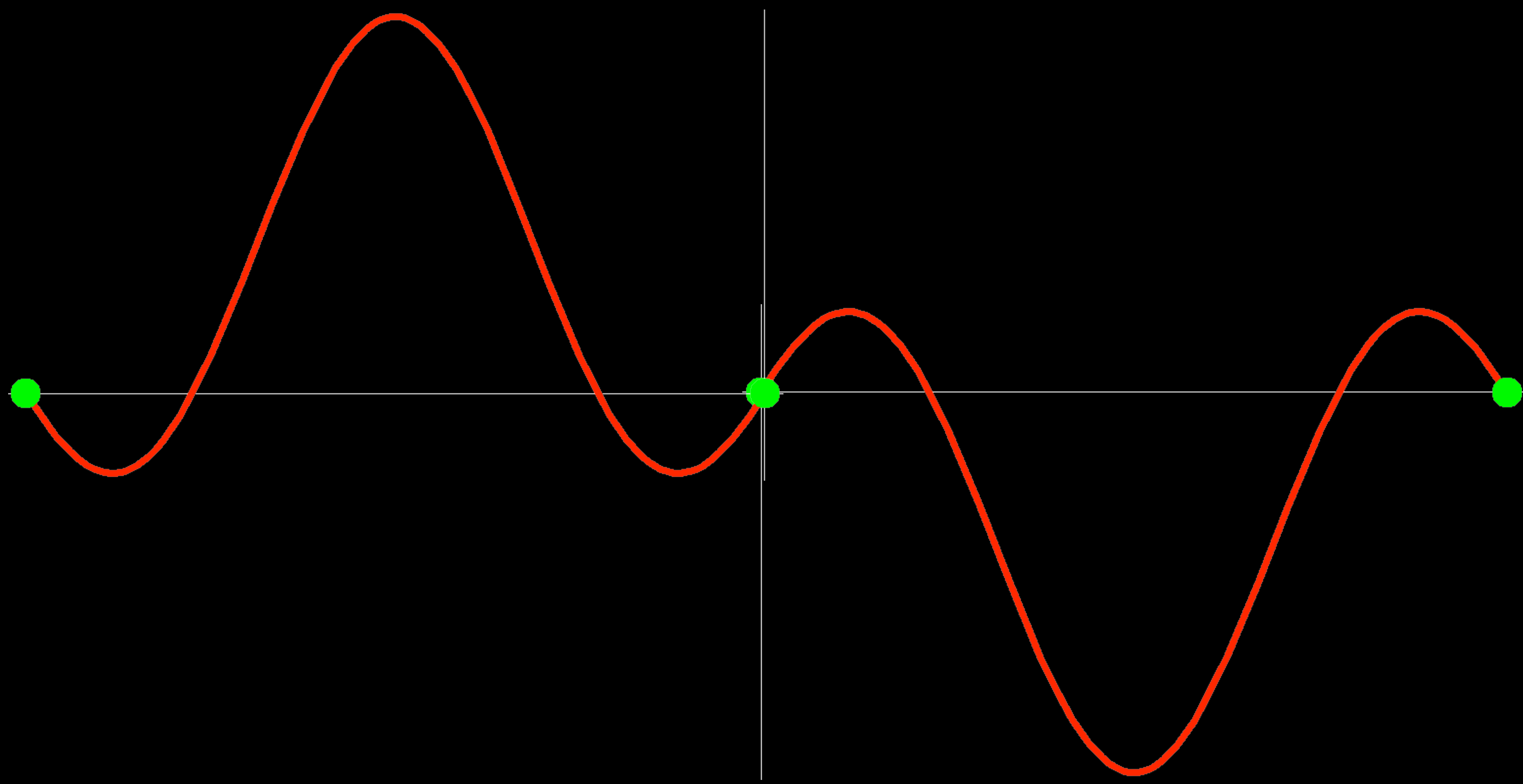


Even Functions



cos- series

Odd Functions

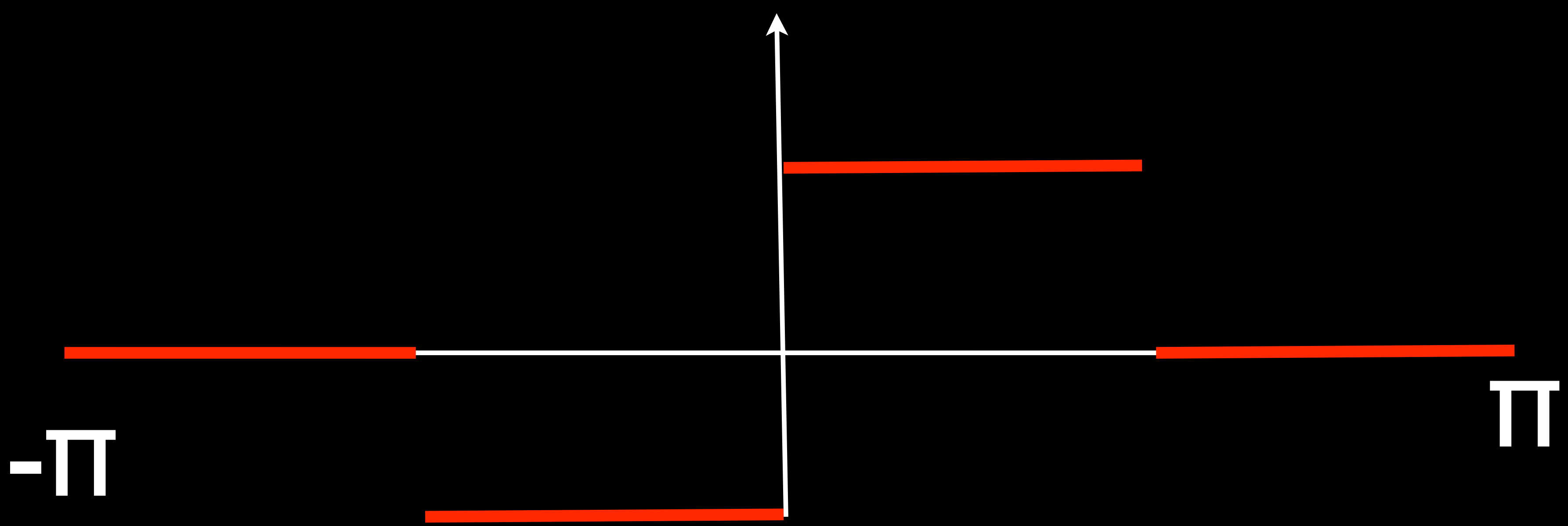


sin- series

problem

Find the Fourier series of the function defined by:

$f(x,0) = \text{sign}(x)$ if $|x|<\pi/2$ and $f(x,0)=0$ if $|x|>\pi/2$.



Parseval Identity

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = ||f||^2$$

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = ||f||^2$$

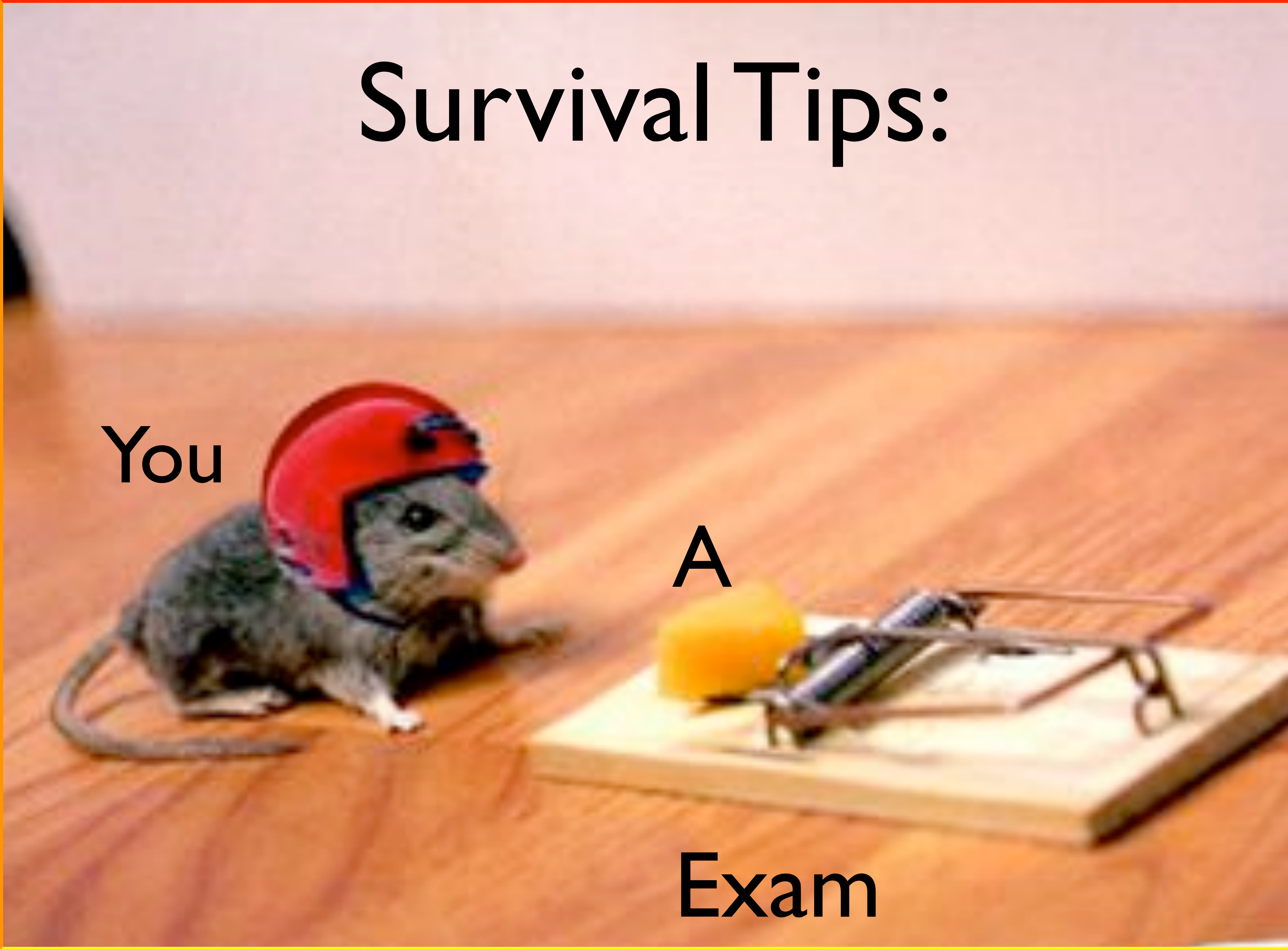
problem

Back to the old problem.
What does Perseval say?

The Fourier coefficients were

$$b_n = \frac{2(1-\cos(n\pi/2))}{n\pi}$$

Survival Tips:



Separate even and odd parts!

Trigonometric polynomials are already Fourier series i.e. sin(5 x) + cos(3 x)

Don’t try to add up the sum. The sum is the final result

Partial differential equations

heat equation: $u_t = u_{xx}$

heat type equation $u_t = p(D^2)\, u$

wave equation: $u_{tt} = u_{xx}$

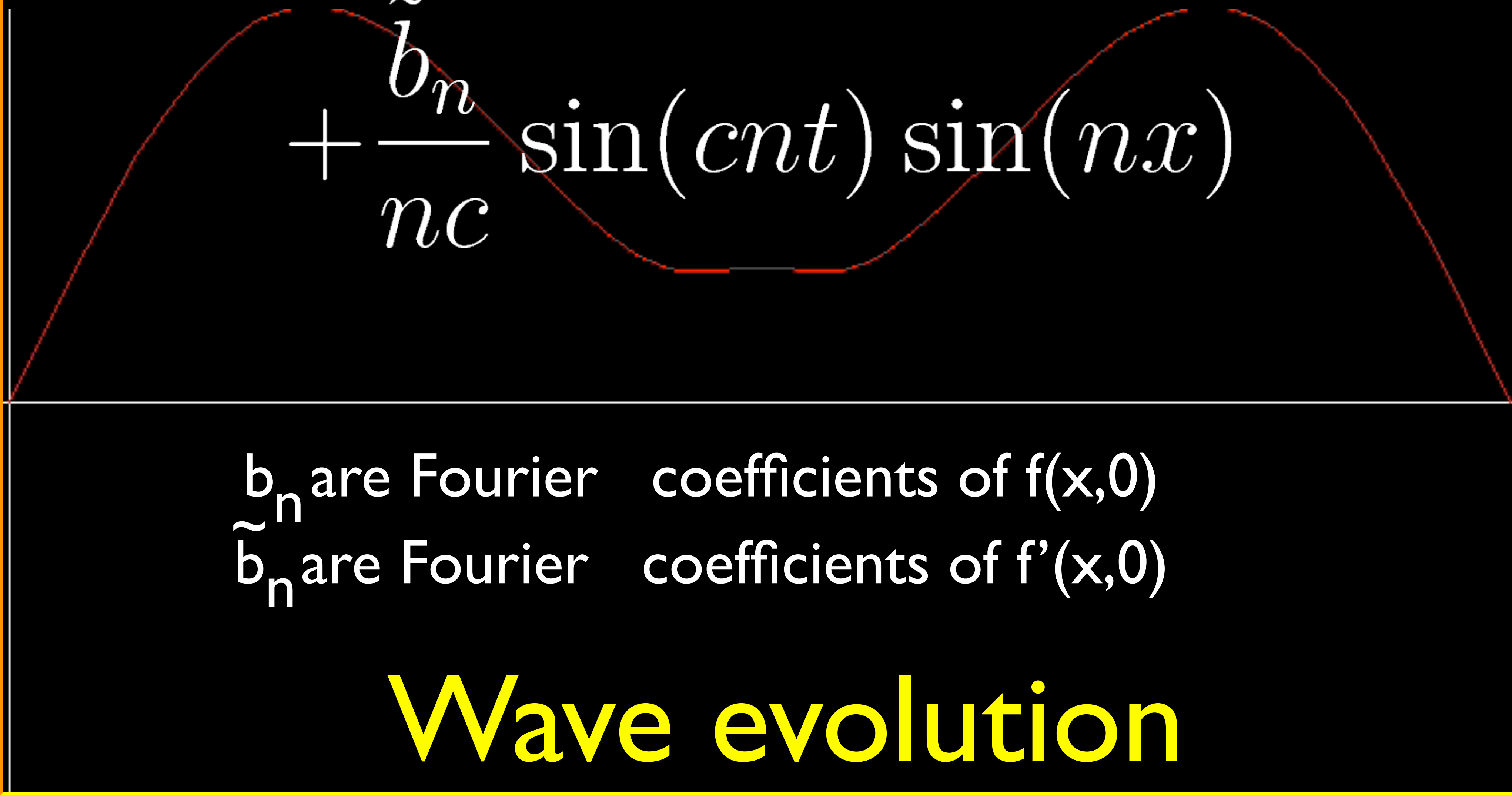
wave type equation: $u_{tt} = p(D^2)\, u$

Heat evolution

$$f(x,t) = \sum_{n \geq 1} b_n e^{-n^2 \mu t} \sin(nx)$$

b_n are Fourier coefficients of f(x,0)

$$f(x,t) = \sum_{n \geq 1} b_n \cos(cnt) \sin(nx) + \frac{\tilde{b}_n}{nc} \sin(cnt) \sin(nx)$$



b_n are Fourier coefficients of f(x,0)

$\tilde{b}_n$ are Fourier coefficients of f’(x,0)

Wave evolution

problem

wave evolution

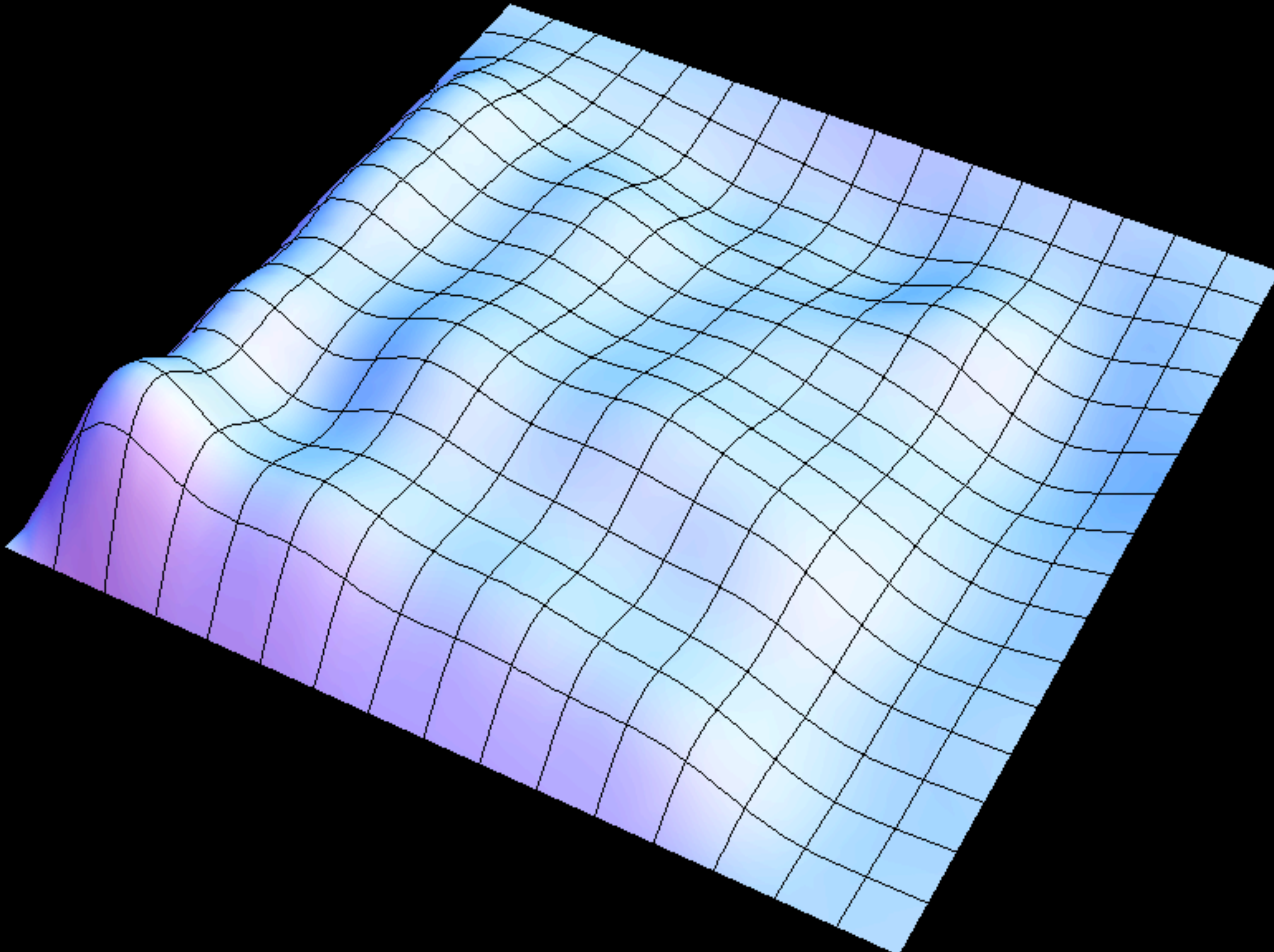
$$u_{tt} = u_{xx} + u_{yy} - u$$

$$u(x,y,0)=\sin(3x)\sin(5y)$$

$$u_t(x,y,0)=\sin(6x)\sin(8y)$$

$$u_{tt} = u_{xx} + u_{yy} - u$$

$u(x,y,0)=\sin(3x)\sin(5y)$ $\quad$ $u_t(x,y,0)=\sin(6x)\sin(8y)$




The end