FIFTH PRACTICE EXAM FINAL **Math 21b, Spring 09**

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Show all your work and justify steps.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If A is an invertible matrix, then A^{-1} and A^T have the same eigenvalues.

2) T F For all $n \geq 0$, $f(x) = \cos(nx)$ is an eigenfunction of $T(f) = f'' - f$.

3) T F All symmetric real matrices are diagonalizable.

4) T F There exists a 3×3 real symmetric matrix which is similar to $B = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

5) T F If A is any matrix, then both AA^T and A^TA are orthogonally diagonalizable.

6) T F All orthogonal projections are diagonalizable

7) T F If the regression line $y = ax + b$ obtained by fitting some data $\{(x_1, y_1), ..., (x_m, y_m)\}$ happens to contain all datapoints, then the corresponding least square solution of $A\vec{x} = \vec{b}$ is an actual solution of $A\vec{x} = \vec{b}$.

8) T F $\det(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}) = -7$.

9) T F There exists a symmetric 2×2 matrix A such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

10) T F The kernel of the operator $T = (D - 2)^5$ is spanned by e^{2t}, te^{2t}, t^2e^{2t}, t^3e^{2t}, t^4e^{2t}.

11) T F Let A be a 2×2 matrix. The system $\frac{dx}{dt} = Ax$ is asymptotically stable if and only if the eigenvalues of A have negative real parts.

12) T F Let A be a 2×2 matrix. $\vec{0}$ is an asymptotically stable equilibrium of $\vec{x}(t+1) = A\vec{x}(t)$ if and only if all eigenvalues of A have negative real parts.

13) T F The subset of $X = C^\infty(\mathbb{R})$, the set of smooth functions of the real line, defined by $Y = \{f \in C^\infty(\mathbb{R}) : f(0) = 1\}$ is a linear subspace of X.

14) T F The subset of $C^\infty(\mathbb{R})$ defined by $Y = \{f \in C^\infty(\mathbb{R}) : f(0) = f''(2)\}$ is a linear subspace of X.

15) T F The differential operator $T(f) = (D^2 + 8D + 17)f$ defined on $C^\infty(\mathbb{R})$ has as the image $C^\infty(\mathbb{R})$.

16) T F If $\vec{0}$ is a least squares solution of $A\vec{x} = \vec{b}$, then $\vec{b} \in \text{im}(A)^\perp$.

17) T F If A is 2×2 matrix with $\det(A) < 0$, then the system $\frac{dx}{dt} = Ax$ has 0 as a stable equilibrium.

18) T F In the Fourier series expansion of the function $f(x) = x + 1$ on $[-\pi, \pi]$, the coefficients a_n belonging to $\cos(nx)$ are zero for all $n \geq 1$.

19) T F $\vec{0}$ is a stable equilibrium of the discrete dynamical system $\vec{x}(t+1) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \vec{x} - 2\vec{x}$.

20) T F If A and B are diagonalizable matrices with the same eigenvectors, then $A + B$ is diagonalizable.

Problem 2) (10 points)

Match the following differential equations with the correct description. Every equation matches exactly one description. No justifications are necessary.

a)
$$\frac{d}{dt}x = 3x - 5y$$
$$\frac{d}{dt}y = 2x - 3y$$

b)
$$\frac{d}{dt}x = -4y + 2x^2 + 2x^3$$
$$\frac{d}{dt}y = 4y(1 - x^2)$$

c)
$$\frac{d}{dt}x = -x + 2y - y^2$$
$$\frac{d}{dt}y = 3x - y - xy - y^2$$

d)
$$\frac{d}{dt}x = 3x - 5y$$
$$\frac{d}{dt}y = x^2 + y^2 + 2$$

e)
$$\frac{d}{dt}x = 2y(x - y) - x$$
$$\frac{d}{dt}y = y(x - y) - y$$

Fill in 1),...,5) here.

a)	b)	c)	d)	e)

1) The equation has a stable equilibrium at $x = 1, y = 1$.
2) The equation has an unstable equilibrium at $x = 1, y = 1$.
3) The equation has a non-constant solution which stays on the line $x = y$.
4) The equation has a closed periodic trajectory.
5) The equation has no equilibria.

Problem 3) (10 points)

a) Find all 2×2 matrices which are both upper triangular and orthogonal.

b) Find a matrix A which has $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the kernel of A.

c) Find a 2×2 matrix A such that $AA^T = A^TA$.

d) Find a matrix B such that $\det(BB^T) \neq \det(B^TB)$.

e) Find a 5×5 matrix A which has 0 as an eigenvalue with geometric multiplicity 3 and the eigenvalue 2 with geometric multiplicity 2.

f) Find a 2×2 matrix which is nondiagonalizable and which has $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as an eigenvector.

Problem 4) (10 points)

Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

a) Find all possibly complex eigenvalues of A with their algebraic multiplicities.
b) Does A have a possibly complex eigenbasis? If so, find one.
c) Is A diagonalizable? Why or why not?
d) Let T be the linear transformation defined by $T(v) = Av$. Describe T geometrically.

Problem 5) (10 points)

Find the function $f(x) = a + b\cos(x)$ which best fits the data

$$\begin{aligned} (x_1, y_1) &= (0, 1) \\ (x_2, y_2) &= (\pi/2, -1) \\ (x_3, y_3) &= (\pi, 1) \\ (x_4, y_4) &= (2\pi, 1) \end{aligned}$$

Problem 6) (10 points)

a) Find the solution of the differential equation $f'(t) + 3f(t) = e^{-2t}, f(0) = 0$.
b) Find the general solution of $f''(t) + 4f'(t) + 3f(t) = 1$.
with $f(0) = 1/3, f(1) = 1/3 + 1/e^3 - 1/e$.
c) Find the solution of $f''(t) = -4f(t)$ with $f(0) = 1, f'(0) = 2$.

Problem 7) (10 points)

a) (7 points) Find a 4×4 matrix A with entries $0, +1$ and -1 for which the determinant is maximal.
b) (3 points) Find the QR decomposition of A.

Problem 8) (10 points)

a) Diagonalize $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

b) Let A be the matrix $\begin{bmatrix} 7 & 4 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & -7 \end{bmatrix}/9$, which has eigenvalues $-1, 0, 1$. Let P be the orthogonal projection onto the kernel of A. Find real numbers a, b, c such that $P = a(A - bI)(A - cI)$.

Hint: b) requires no calculations, rather thought).

Problem 9) (10 points)

Define $f = \sinh(x) = \frac{e^x - e^{-x}}{2}$ on $C^\infty([0, \pi])$ be a function on the interval $[0, \pi]$. Find a solution $u(x, t)$ of the heat equation $u_t = u_{xx}$ which satisfies $u(0, x) = f(x)$.

Hint. $\int \sinh(x) \sin(nx)\, dx = \frac{\cosh(x)\sin(nx) - n\cos(nx)\sinh(x)}{1+n^2}$. You can leave terms like $\sinh(\pi)$.

Problem 10) (10 points)

a) Find the Fourier series of $|\sin(x/2)|$ on $C([-\pi, \pi])$.

b) Find $\sum_{n=1}^{\infty} (-1)^n \frac{1}{4n^2-1}$.

Problem 11) (10 points)

An ecological system consists of two species whose populations at time t are given by $x(t)$ and $y(t)$. The evolution of the system is described by the equation

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(x-y+1) \\ y(x+y-3) \end{bmatrix} .$$

a) Find all equilibrium points and nullclines of this system in $x \geq 0, y \geq 0$.
b) Sketch the vector field of this system in the first quadrant $x \geq 0, y \geq 0$ indicating the direction of the vector field along the nullclines and inside the regions determined by the nullclines.
c) Are there any stable equilibrium points? Justify your answers.
d) If both species start with positive populations, can either become extinct? Explain.

Problem 12) (10 points)

Consider the linear differential equation

$$\begin{aligned} \frac{dx}{dt} &= ax + y \\ \frac{dy}{dt} &= ay \\ \frac{dz}{dt} &= -z \, . \end{aligned}$$

a) Write the system in the form $\frac{d}{dt}\vec{x} = A\vec{x}$, where A is a matrix.
b) For which parameters a is the system stable?

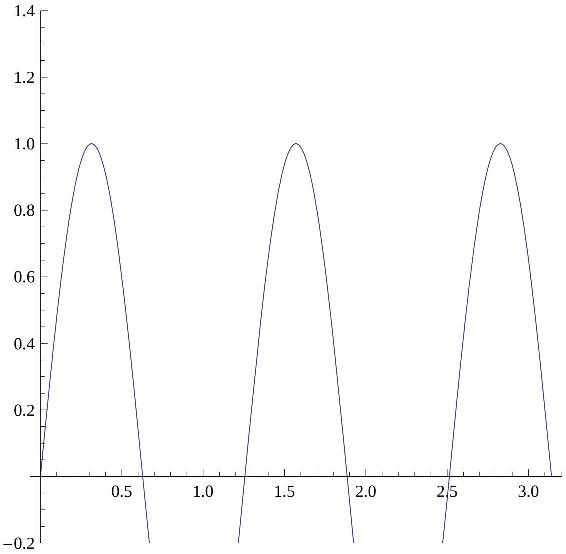
Problem 13) (10 points)

Find the solution of the modified wave equation $f_{tt} = 16 f_{xx} + f$ with initial string position

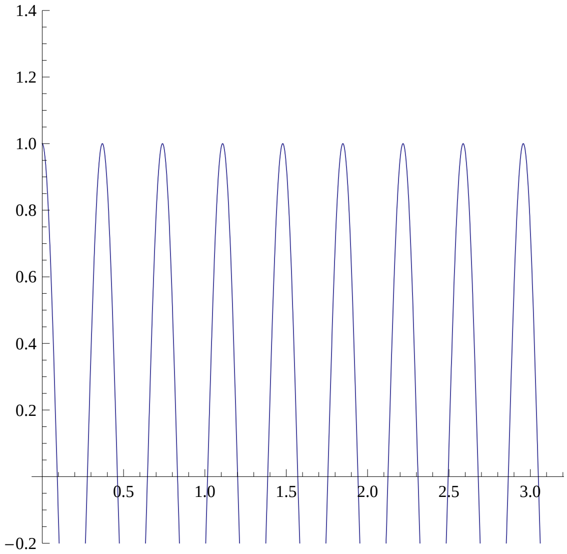
$$f(x, 0) = \sin(5x)$$

and initial string velocity

$$f_t(x, 0) = \sin(17x) \, .$$



Initial wave position on $[0, \pi]$.



Initial wave velocity on $[0, \pi]$.

Problem 14) (10 points)

Because a higher wave is accelerated more, there is a force proportional to u and the partial differential equation is

$$u_{tt} = u_{xx} + 5u \; .$$

a) (3 points) Solve the system for initial condition $u(x, 0) = \sin(3x) - \sin(7x)$ and $u_t(x, 0) = 0$.
b) (3 points) Monster waves: Find the solution of the differential equation given in b), if the initial condition is $\sin(x)$.
c) (4 points) Verify that the wave height of the system

$$u_{tt} = cu_{xx} + bu$$

stays bounded at all times if the wind strength b is small enough or the string constant c is big enough. Verify that for $b > c$, we have solutions which explode as in b).