1. The verification that $\cos(nx), \sin(nx), 1/\sqrt{2}$ form an orthonormal family is a straightforward computation when using the identities provided. For example, $\langle \cos(nx), \sin(mx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) \sin(mx) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((n-m)x) - \cos((n+m)x) \, dx$ which is equal to 1 if $n = m$ and equal to 0 if $n \neq m$. The computations can be abbreviated by noting that integrating an odd 2π periodic function over $[-\pi, \pi]$ is zero.

2. To get the Fourier series of the function $f(x) = |x|$, note first that this is an **even function** so that it has a cos series. We compute

$$a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{2}{\pi} \int_0^\pi x \frac{1}{\sqrt{2}} \, dx = \frac{\pi\sqrt{2}}{2} \ .$$

$$a_n = \langle f, \cos(nx) \rangle = \frac{2}{\pi} \int_0^\pi x \cos(nx) \, dx = \frac{2}{\pi} \left[\frac{\cos(n\pi) - 1}{n^2} \right] \ .$$

3. The Fourier series of $\cos^2(x) + 5\sin(7x) + 55$ is with

$$\cos^2(x) = \frac{\cos(2x) + 1}{2}$$

given as $\boxed{1/2 + 55 + \cos(2x)/2 + 5\sin(7x)}$. All Fourier coefficients are zero except $\boxed{a_0 = (55 + 1/2)\sqrt{2} \text{ and } a_2 = 1/2 \text{ and } b_7 = 5}$.

4. To find the Fourier series of the function $f(x) = |\sin(x)|$, we first note that this is an **even function** so that it has a cos-series. If we integrate from 0 to π and multiply the result by 2, we can take the function $\sin(x)$ instead of $|\sin(x)|$ so that

$$a_n = \frac{2}{\pi} \int_0^\pi \sin(x) \cos(nx) \, dx \ .$$

We use one of the trigonometric identities provided in the text to solve this integral.

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos(2x)}{2^2 - 1} + \frac{\cos(4x)}{4^2 - 1} + \frac{\cos(6x)}{6^2 - 1} + ... \right)$$

5. The square of the length of the function $f(x)$ is 1. Parceval's identity shows that

$$1 = a_0^2 + \sum_{n=1}^\infty a_n^2 = (\sqrt{2}\frac{2}{\pi})^2 + \frac{16}{\pi^2} \left[\frac{1}{(2^2 - 1)^2} + \frac{1}{(4^2 - 1)^2} + \frac{1}{(6^2 - 1)^2} + \cdots \right]$$

so that the sum is $\boxed{\pi^2/16 - 1/2}$.

6. To solve the heat equation $f_t = \mu f_{xx}$ on $[0, \pi]$ with the initial condition $f(x, 0) = |\sin(3x)|$, we make a Fourier expansion of $|\sin(3x)|$:

$$|\sin(3x)| = \sum_{n=1}^\infty b_n \sin(nx)$$

and can immediately write down the solution

$$f(x, t) = \sum_{n=1}^\infty b_n e^{-\mu n^2 t} \sin(nx) \ .$$

Now to the Fourier series: note that $\sin(3x)$ is nonnegative on $[0, \pi/3]$ and $[2\pi/3, \pi]$ so that it agrees there with the function $|\sin(3x)|$. On the interval $[\pi/3, 2\pi/3]$ however the function $\sin(3x)$ is negative or zero so that $-\sin(3x)$ is nonnegative there. We get therefore the Fourier coefficients as

$$b_n = \frac{2}{\pi} \left[\int_0^{\pi/3} \sin(3x) \sin(nx) \, dx - \int_{\pi/3}^{2\pi/3} \sin(3x) \sin(nx) \, dx + \int_{2\pi/3}^\pi \sin(3x) \sin(nx) \, dx \right] .$$

We use the identity

$$2\sin(nx)\sin(my) = \cos(nx - my) - \cos(nx + my)$$

to solve these integrals:

$$b_n = \frac{2}{\pi} \left[\left(\frac{-3\sin\left(\frac{n\pi}{3}\right)}{n^2 - 9} \right) - \left(\frac{3\sin\left(\frac{n\pi}{3}\right)}{n^2 - 9} + \frac{3\sin\left(\frac{2n\pi}{3}\right)}{n^2 - 9} \right) + \left(\frac{-3\sin\left(\frac{2n\pi}{3}\right)}{n^2 - 9} \right) \right]$$

which is

$$\boxed{\frac{2}{\pi} \left[\frac{-3\sin\left(\frac{n\pi}{3}\right)}{n^2 - 9} \right]} \ .$$

The case $n = 3$ might look problematic at first, but the limit still exists and $b_3 = \pi/6$.

7. Because the function $h(x, t) = (b - a)x/\pi + a$ does not depend on t and the second x-derivative is zero, it is a solution of the heat equation. Every function which is linear in x and does not depend on t is a solution to the heat equation.

8. The crucial fact to note here is the **linearity** of the heat equation: solutions to a heat equation can be added and scaled and also the resulting sum or scaled function remains a solution. The functions f and h both solve the heat equation and have the same boundary conditions. Their difference $g = f - h$ also solves the heat equation and has the boundary condition 0 at both ends $x = 0$ and $x = \pi$.

9. So solve the heat equation with the initial condition $f(x, 0) = f(x) = \sin(5x) + 3\sin(x) + x/\pi$ and satisfying $f(0, t) = 0, f(\pi, t) = 1$, we subtract the special solution x/π so that we have the problem to find the Fourier development of the function $g(x) = \sin(5x) + 3\sin(x)$ which is trivially already given as $\sin(5x) + 3\sin(x)$, where $b_1 = 3, b_5 = 1$ and $b_n = 0$ for all $n \neq 1, 5$. since we know the solution of $g(x, t) = e^{-25t}\sin(5x) + 3e^{-t}\sin(t)$ we have the final solution $f(x, t) = g(x, t) + h(x, t)$ which is

$$\boxed{f(x, t) = x/\pi + e^{-25t}\sin(5x) + 3e^{-t}\sin(t)} \ .$$

10. Because the initial condition is zero on the interval $[\pi/2, \pi]$, we have to integrate from 0 to $\pi/2$ only. The Fourier coefficients of the function $g(x)$ can be computed using one of the trigonometric identities

$$\frac{2}{\pi} \int_0^{\pi/2} \sin(2x) \sin(nx) = \frac{4(-1)^n}{\pi(n^2 - 4)} \sin(n\pi/2) \ .$$

The Fourier series of the initial position $f(x) = 0$ of the string is equal to zero by assumption. The solution of the wave equation is

$$f(x, t) = \sum_{n=1}^\infty \frac{4(-1)^n}{\pi(n^2 - 4)} \sin(n\pi/2) \sin(nx) \sin(nt) \frac{1}{n} \ .$$